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I.—ON THE RESIDUAL CALCULUS.

CAUCHY, in several articles in his "*Exercices*," has developed a new Calculus, to which he has given the name of the "*Calcul des Résidus.*" Few persons appear to have followed the author in the use of it; and in our own language I do not recollect having met even with an allusion to it: yet it deserves some attention at the hands of mathematicians, were it only from the deserved celebrity of its author. I propose to give here merely a slight sketch of its principles, with a few examples; and if any one should be induced from this specimen to wish for a more complete knowledge of this Calculus, I must refer him to the Memoirs of its author.

The Residual Calculus bears a certain analogy to the Differential Calculus, for in both the object is to investigate the nature and properties of certain functions which appear in an indeterminate form, but yet have finite values.

Let $f(x)$ be a function of x, which becomes infinite when $x = a$; that is, let a be a root of the equation

$$f(x) = \infty, \text{ or } \frac{1}{f(x)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then the expression $(x-a)f(x)$ becomes indeterminate when $x = a$, as it takes the form $0 \times \infty$, though it may still have really a finite value. Suppose this to be the case, and that it is a certain function of a which we may call $\mathrm{R}(a)$. Then we have

$$(x-a)f(x) = \mathrm{R}(x) \text{ when } x = a \quad \ldots\ldots\ldots\ldots (2),$$

which gives $$f(x) = \frac{\mathrm{R}(x)}{x-a} \text{ when } x = a \quad \ldots\ldots\ldots\ldots (3).$$

This function $\mathrm{R}(a)$ is called the residue of $f(x)$ with respect to a. The operation of finding the residue is called the extraction of the residue, and is represented by the characteristic symbol ς, so that we write

$$\mathrm{R}(a) = \varsigma f(x).$$

In equation (3) let us suppose $x = a + h$. Then

$$f(a + h) = \frac{\mathrm{R}(a + h)}{h} = \frac{1}{h}\{\mathrm{R}(a) + h\,\mathrm{R}'(a) + \&c.\},$$

when $x = a$, or $h = 0$.

Whence we find the residue $\mathrm{R}(a)$ of $f(x)$ is the coefficient of $\frac{1}{h}$ in the expansion of $f(a + h)$, where a is a root of the equation $f(x) = \infty$.

We have spoken only of one root of this equation, but there may be several, and for each of these there is a corresponding residue. The sum of the residues corresponding to each root of the equation is called the *integral* residue of $f(x)$. To distinguish the integral from the partial residues, I shall suffix the root to the Residual Symbol in the latter case.* Thus, $\varsigma f(x)$ will represent the integral residue of $f(x)$, $\varsigma_a f(x)$, $\varsigma_b f(x)$, $\varsigma_c f(x)$, &c. will represent the partial residues with respect to a, b, c, &c. these being roots of the equation $f(x) = \infty$.

Let us suppose, now, that instead of the expression

$$(x - a) f(x)$$

having a finite value when $x = a$, that it is the expression

$$(x - a)^m f(x)$$

which has a finite value when $x = a$; in which case $f(x) = \infty$ is said to have m equal roots. If, as before, we suppose $\mathrm{R}(a)$ to be the finite value, we shall have

$$f(x) = \frac{\mathrm{R}(x)}{(x - a)^m} \text{ when } x = a,$$

or putting $x = a + h$, and expanding $\mathrm{R}(a + h)$ by Taylor's theorem,

$$f(a + h) = \frac{1}{h^m}\left\{\mathrm{R}(a) + \frac{h}{1}\mathrm{R}'(a) + \&c. + \frac{h^{(m-1)}\mathrm{R}^{(m-1)}(a)}{1.2 \ldots (m - 1)} + \&c.\right\}$$

The coefficient of $\frac{1}{h}$ in this expansion is

$$\frac{\mathrm{R}^{(m-1)}(a)}{1.2 \ldots (m - 1)},$$

which is the residue of the function $f(x)$ corresponding to $x = a$.

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* This is not Cauchy's notation.

Since $\mathrm{R}(x) = h^m f(x) = h^m f(a+h)$,

$$\mathrm{R}^{(m-1)}(a) = \frac{d^{m-1}}{dh^{m-1}} \{h^m f(a+h)\} \text{ when } h = 0;$$

and therefore we have

$$(4)\ \varepsilon_a f(x) = \frac{1}{1.2...(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{h^m f(a+h)\} \text{ when } h = 0.$$

I shall now proceed to show how the extraction of residues may be simplified when $f(x)$ has particular forms: but previously it is necessary to explain the following notation. When $f(x)$ consists of distinct factors, which are functions of x, the roots of $f(x) = \infty$ may be the roots of any one factor equated to ∞.

When then we wish to indicate that the extraction of residues has reference only to the roots of one of the factors, that factor is enclosed in brackets [ ]. As, for instance, if $f(x) = \phi(x).\psi(x)$. and we wish to represent the extraction of the residues with reference to the roots of $\phi(x) = \infty$, we shall write

$$\varepsilon\,[\phi(x)].\psi(x).$$

If a factor is of the form $\frac{1}{\phi(x)}$, we should properly represent the extraction of the residues with respect to it by

$$\varepsilon\left[\frac{1}{\phi(x)}\right].\psi(x).$$

But since $\frac{1}{\varphi(x)} = \infty$ gives the same roots as $\phi(x) = 0$, which, therefore, is the equation to which we look, we shall find it convenient to use the notation

$$\varepsilon\,\frac{\psi(x)}{[\phi(x)]}.$$

Hence, if a be a root of $f(x) = \infty$, the residue with regard to it may be expressed by $\varepsilon\,\frac{(x-a)f(x)}{[x-a]}$ instead of $\varepsilon_a f(x)$, so that

$$\varepsilon\,\frac{(x-a)f(x)}{[x-a]} = hf(a+h) \text{ when } h = 0,$$

if there be only one root of $f(x) = \infty$; and

$$\varepsilon\,\frac{(x-a)f(x)}{[x-a]} = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{h^m f(a+h)\} \text{ when } h=0,$$

if there be m roots equal to a.

Now suppose $f(x)$ to be of the form $\frac{\mathrm{f}(x)}{\mathrm{F}(x)}$, and that a is a root of $\mathrm{F}(x) = 0$, then, as $\mathrm{F}(a) = 0$, we have

$$\mathrm{F}(a+h) = h\,\mathrm{F}(a+\theta h),$$

and therefore

$$hf(a+h) = \frac{h\,\mathrm{f}(a+h)}{h\,\mathrm{F}'(a+\theta h)}.$$

So that when $h = 0$,

$$\mathfrak{E}_a \frac{\mathrm{f}(x)}{[\mathrm{F}(x)]} = \frac{\mathrm{f}(a)}{\mathrm{F}'(a)} \quad \ldots\ldots\ldots\ldots (5).$$

As an example, take $\frac{x^m}{x-a}$, whence

$$\mathfrak{E}_a \frac{x^m}{[x-a]} = a^m.$$

Also, $$\mathfrak{E}_a \frac{x^2 - px + q}{[x^2 - a^2]} = \frac{a^2 - pa + q}{2a},$$

$$\mathfrak{E}_a \frac{x^m}{[(x-a)(x-b)]} = \frac{a^m}{a-b}.$$

If the fraction be of the form $\frac{\mathrm{f}(x)}{(x-a)(x-b)\ldots(x-r)}$, where there are r factors in the denominator, we may extract the residue with respect to each factor, and the sum of all these partial residues will be the integral residue with respect to the denominator. Hence

$$\mathfrak{E} \frac{\mathrm{f}(x)}{[(x-a)(x-b)\ldots(x-r)]} = \frac{\mathrm{f}(a)}{(a-b)(a-c)\ldots(a-r)}$$

$$+ \frac{\mathrm{f}(b)}{(b-a)(b-c)\ldots(b-r)} + \&c. + \frac{\mathrm{f}(r)}{(r-a)(r-b)\ldots(r-q)}.$$

When the denominator has m roots, each equal to a,

$$\mathfrak{E}_a f(x) = \mathfrak{E}_a \frac{\mathrm{f}(x)}{[\mathrm{F}(x)]} = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{h^m f(a+h)\}$$

when $h = 0$.

But $f(a+h) = \frac{\mathrm{f}(a+h)}{\mathrm{F}(a+h)}$, and

$$\mathrm{F}(a+h) = \mathrm{F}(a) + \mathrm{F}'(a)\frac{h}{1} + \&c. + \frac{\mathrm{F}^{(m-1)}(a)h^{m-1}}{1.2\ldots(m-1)} + \frac{\mathrm{F}^{(m)}(a)h^m}{1.2\ldots m} + \&c.$$

and as these are m equal roots of $\mathrm{F}(x) = 0$, we have

$$\mathrm{F}(a) = 0, \quad \mathrm{F}'(a) = 0 \ \ldots\ \mathrm{F}^{(m-1)}(a) = 0.$$

Hence $$\mathrm{F}(a+h) = \frac{h^m}{1.2\ldots m}\{\mathrm{F}^{(m)}(a) + \mathrm{F}^{(m+1)}(a)\frac{h}{m+1} + \&c.\},$$

or $$\mathrm{F}(a+h) = \int^m dh^m \frac{d^m}{da^m} \mathrm{F}(a+h);^*$$

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* By the notation $\frac{d^m}{da^m} f(a+h)$, I mean to express the value which $\frac{d^m}{dx^m} f(x+h)$ takes when $x = a$.

whence we find

$$\mathfrak{E}_a \frac{f(x)}{[F(x)]} = \frac{1}{1.2...(m-1)} \frac{d^{m-1}}{dh^{m-1}} \left\{ \frac{h^m f(a+h)}{\int^m dh^m \frac{d^m}{da^m} F(a+h)} \right\}$$

when $h=0$ ..... (6).

When $F(a+h)$ is of the form $(x-a)^m$, this expression is considerably simplified. For

$$\frac{d^m}{dx^m}(x-a)^m = m(m-1)...2.1,$$

$$\text{and } \int^m dh^m \frac{d^m}{da^m} F(a+h) = h^m, \text{ so that}$$

$$\mathfrak{E}_a \frac{f(x)}{(x-a)^m} = \frac{1}{1.2...(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{f(a+h)\} \text{ when } h=0 .....(7).$$

$$\text{Thus } \mathfrak{E}_a \frac{x^n}{(x-a)^4} = \frac{n.(n-1)(n-2)}{1.2.3} a^{n-3}.$$

I shall now proceed to the proof of one of the fundamental theorems of the Residual Calculus. It is, that

$$f(x) - \mathfrak{E} \frac{f(z)}{x-z} = \psi(x) ..... (8),$$

$\psi(x)$ being a function of x, which takes generally a finite value when

$$x = a_1, \ = a_2, \ = a_3, \text{ \&c.},$$

a_1, a_2, a_3, being the roots of the equation $f(x) = \infty$.

We have generally

$$f(x) = \frac{R(x)}{(x-a_1)^m} \text{ when } x = a_1.$$

Expanding $R(x) = R(a_1 + h)$ by Taylor's theorem, and putting $x - a_1$ for h,

$$f(x) = \frac{R(a_1)}{(x-a_1)^m} + \frac{1}{1} \frac{R'(a_1)}{(x-a_1)^{m-1}} + \text{\&c.}$$

$$+ \frac{1}{1.2...(m-1)} \frac{R^{(m-1)}(a_1)}{x-a_1} + \psi(x) ..... (9),$$

where $\psi(x)$ is a function of x which generally takes for $x = a_1$ the finite value $\frac{R^{(m)}(a_1)}{1.2...m}$.

But we have also from the definition of a residue

$$\frac{R^{(m)}(a_1)}{1.2...m} = \mathfrak{E}_{a_1} \frac{R(z)}{(z-a_1)^{m+1}} ..... (10),$$

and similarly for the others. Hence

$$f(x) = \frac{1}{(x-a_1)^m}\left\{\mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{z-a_1} + \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(z-a_1)^2}(x-a_1) + \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(z-a_1)^3}(x-a)^2 + \&c.\right\}$$

$$= \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(x-a_1)^m(z-a_1)^m}\left\{(z-a_1)^{m-1} + (z-a_1)^{m-2}(x-a_1) + \&c.\right\} + \psi(x)$$

$$= \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(x-a_1)^m(z-a_1)^m}\left\{\frac{(x-a_1)^m - (z-a_1)^m}{x-z}\right\} + \psi(x)$$

$$= \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(x-z)(z-a_1)^m} - \frac{1}{(x-a_1)^m}\mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{x-z} + \psi(x).$$

Now the second term of this expression is equal to 0, because $\frac{\mathrm{R}(z)}{x-z}$ does not become infinite for $z = a_1$, and consequently has no corresponding residue. We have therefore

$$f(x) = \mathfrak{E}_{a_1}\frac{\mathrm{R}(z)}{(x-z)(z-a_1)^m} + \psi(x) = \mathfrak{E}_{a_1}\frac{f(z)}{x-z} + \psi(x),$$

since $\mathrm{R}(z) = (z-a_1)^m f(z)$.

Hence, in order to deduce from $f(x)$, which becomes infinite for $x = a_1$, a function of x, which shall remain finite under the same circumstances, we must subtract from $f(x)$ the partial residue with respect to a_1 of $\frac{f(z)}{x-z}$. Similarly, if we wish to find a function of x which shall not become infinite when $x = a_1$, $= a_2$, &c we must subtract from $f(x)$ the partial residues with respect to each of these roots. So that if we make

$$f(x) - \mathfrak{E}\frac{[f(z)]}{x-z} = \phi(x),$$

$\phi(x)$ will have a finite value when $x = a_1$, $= a_2$, $= a_3$, &c.

The notation $\mathfrak{E}\frac{[f(z)]}{x-z}$ it must be remembered, means the sum of all the residues with respect to the roots of the equation $f(z) = \infty$.

From this theorem we can deduce a number of important results. If $f(x)$ is a fraction, as $\frac{\mathrm{f}(x)}{\mathrm{F}(x)}$, $\varphi(x)$ must be a fraction of the same kind, the denominator of which must never become 0, since the fraction $\phi(x)$ can never become infinite.

Therefore the denominator must be independent of x or constant, and $\phi(x)$ an integral function of x. If $\mathrm{F}(x)$ be of higher dimensions than $\mathrm{f}(x)$, $f(x)$ will vanish for $x = \infty$, as likewise will the residue, so that we must also have $\phi(x) = 0$. Hence in this case

$$f(x) = \mathfrak{E}\frac{[f(z)]}{x-z} \quad \ldots\ldots\ldots (11),$$

which gives the means of decomposing a rational fraction unto simple fractions.

As an example, take the fraction $\frac{x^2-7x+1}{(x-1)(x-2)(x-3)} = f(x)$.

$$f(x) = \mathfrak{E}\frac{z^2-7z+1}{(x-z)[(z-1)(z-2)(z-3)]} =$$

$$= \mathfrak{E}_1\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)} + \mathfrak{E}_2\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)}$$

$$+ \mathfrak{E}_3\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)}$$

$$= -\frac{5}{2}\frac{1}{x-1} + 9\frac{1}{x-2} - \frac{11}{2}\frac{1}{x-3}.$$

Again, take $\frac{1}{(x-1)^2(x+1)} = f(x)$

$$f(x) = \mathfrak{E}_{-1}\frac{1}{(x-z)(z-1)^2(z+1)} + \mathfrak{E}_1\frac{1}{(x-z)(z-1)^2(z+1)}$$

$$= \tfrac{1}{4}\frac{1}{x+1} + \frac{d}{dh}\frac{1}{(x-1-h)(2+h)} \text{ when } h = 0$$

$$= \tfrac{1}{4}\frac{1}{x+1} + \tfrac{1}{2}\frac{1}{(x-1)^2} - \tfrac{1}{4}\frac{1}{x-1}.$$

When the fraction is of the form $\frac{f(x)}{(x-a)^m}$, we can easily deduce by this method the series of component simple fractions.

For $\frac{f(x)}{(x-a)^m} = \mathfrak{E}_a\frac{f(z)}{(x-z)(z-a)^m}$

$$= \frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}}{dh^{m-1}}\left(\frac{f(a+h)}{x-a-h}\right) \text{ when } h = 0;$$

and if we actually perform the operations indicated, we get

$$\frac{f(x)}{(x-a)^m} = \frac{1}{1.2\ldots(m-1)}\left\{\frac{f^{(m-1)}(a)}{(x-a)} + (m-1)\frac{f^{(m-2)}(a)}{(x-a)^2}\right.$$

$$+ (m-1)(m-2)\frac{f^{(m-3)}(a)}{(x-a)^2} + \&c.$$

$$\left. + (m-1)(m-2)\ldots 2.1\frac{f(a)}{(x-a)^m}\right\} \ldots\ldots\ldots (12).$$

This also appears from (9) by inverting the order of the terms.

By the application of the theorem (11) we can easily prove Lagrange's formula for interpolation. For if

$$f(x) = \frac{f(x)}{(x-x_1)(x-x_2)\ldots(x-x_m)},$$

we have

$$\mathcal{E}\frac{f(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$
$$=\mathcal{E}_{x_1}\frac{f(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$
$$+\mathcal{E}_{x_2}\frac{f(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}+\&c.$$
$$+\mathcal{E}_{x_m}\frac{f(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$
$$=\frac{f(x_1)}{(x_1-x_2)\ldots(x_1-x_m)}\frac{1}{x-x_1}+\&c.$$
$$+\frac{f(x_m)}{(x_m-x_1)\ldots x_m-x_{m-1}}\frac{1}{x-x_m};$$

whence $f(x)=\dfrac{(x-x_2)\ldots(x-x_m)}{(x_1-x_2)\ldots(x_1-x_m)}f(x_1)+\&c.$

$$+\frac{(x-x_1)\ldots x-x_{m-1}}{(x_m-x_1)\ldots x_m-x_{m-1}}f(x_m).$$

Again, taking the equation

$$f(x)=\mathcal{E}\frac{[f(z)]}{x-z},$$

and multiplying both sides by x, it becomes

$$xf(x)=\mathcal{E}\frac{[f(z)]}{1-\frac{z}{x}}.$$

On making $x=\infty$, $f(x)=0$, as the denominator of $f(x)$ is supposed to be of higher dimensions than the numerator; therefore $xf(x)$ may have a finite value. Let this be V, then

$$\mathcal{E}f(z)=V \ldots\ldots\ldots\ldots (13).$$

If $V=0$, which will be the case when the degree of the denominator of $f(x)$ surpasses that of the numerator by more than unity, then we have simply

$$\mathcal{E}f(z)=0 \ldots\ldots\ldots\ldots (14).$$

Let $f(x)=\dfrac{x^n}{(x-x_1)(x-x_2)\ldots(x-x_m)}$,

we find that

$$\frac{x_1^n}{(x_1-x_2)\ldots(x_1-x_m)}+\frac{x_2^n}{(x_2-x_1)\ldots(x_2-x_m)}+\&c.$$
$$+\frac{x_m^n}{(x_m-x_1)\ldots(x_m-x_{m-1})}=1 \text{ or } =0,$$

according as $n=m-1$, or as $n<m-1$; since in the former

case we have $xf(x) = 1$ when $x = \infty$, and in the latter $xf(x) = 0$ when $x = \infty$.

We have as yet limited ourselves to the consideration of proper fractions, but the Residual Calculus may be extended to all others. To do this, we must first prove the following Theorem,

$$\mathfrak{E}f(z) = \mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]} \ldots\ldots\ldots\ldots\ldots(15).$$

We know by (8) that

$$\frac{f\left(\frac{1}{u}\right)}{u^2} = \mathfrak{E}\frac{f\left(\frac{1}{s}\right)}{[s^2](u-s)} + \mathrm{U},$$

where U is a function of u, which remains finite for $u = 0$.

Let there be m roots of $f\left(\frac{1}{s}\right) = \infty$, each equal to 0, and let us expand $\frac{1}{u-s}$; then

$$\frac{f\left(\frac{1}{u}\right)}{u^2} = \frac{1}{u}\mathfrak{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} + \frac{1}{u^2}\mathfrak{E}\frac{sf\left(\frac{1}{s}\right)}{[s^2]} + \&c. + \frac{1}{u^{m+2}}\mathfrak{E}\frac{s^{m+1}f\left(\frac{1}{s}\right)}{[s^2]} + \mathrm{U}.$$

Multiplying by u^2, substituting z for $\frac{1}{u}$, and making $\mathrm{U}u^2 = \phi(z)$,

$$f(z) = \frac{1}{z}\mathfrak{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} + \mathfrak{E}\frac{sf\left(\frac{1}{s}\right)}{[s^2]} + \&c. + z^m\mathfrak{E}\frac{s^{m+1}f\left(\frac{1}{s}\right)}{[s^2]} + \phi(z).$$

Now, extracting the residues with regard to z, and observing that

$$\mathfrak{E}\left(\frac{1}{z}\right) = 1, \quad \mathfrak{E}\,1 = 0, \quad \mathfrak{E}\,z = 0 \ldots \mathfrak{E}\,z^m = 0,$$

we find $$\mathfrak{E}f(z) = \mathfrak{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} + \mathfrak{E}\,\phi(z) \ldots\ldots\ldots(16).$$

Now $\phi(z) = \mathrm{U}u^2$, and U has a finite value for $u = 0$, therefore $\mathrm{U}u = z\phi(z)$ must vanish. Hence by (14)

$$\mathfrak{E}\,\phi(z) = z\,\phi(z) = 0.$$

Hence, finally, we have

$$\mathfrak{E}f(z) = \mathfrak{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} = \mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]} \ldots\ldots\ldots(17).$$

In this last formula substitute $\frac{f(z)}{z-x}$ for $f(z)$. As the residue is taken with regard to all the roots of $\frac{f(z)}{z-x}=\infty$, and one of these is $x=z$, we may separate that partial residue from the others. Thus we have

$$\mathfrak{E}\frac{f(z)}{z-x}=\mathfrak{E}_x\frac{f(z)}{z-x}+\mathfrak{E}\frac{[f(z)]}{z-x}$$

$$=f(x)+\mathfrak{E}\frac{[f(z)]}{z-x};$$

so that, by the last theorem,

$$f(x)+\mathfrak{E}\frac{[f(z)]}{z-x}=\mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]\left(\frac{1}{z}-x\right)}=\mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}$$

$$\text{or } f(x)=\mathfrak{E}\frac{[f(z)]}{x-z}+\mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)} \quad \ldots\ldots\ldots(18),$$

which is the equation we ought to substitute for the formula (11), if $f(z)$ becomes infinite when $z=\infty$.

If $f(x)$ is an integral function of x, the residue $\mathfrak{E}\frac{[f(z)]}{x-z}$ vanishes, since $f(z)$ has a finite value for any value of z, so that the expression is reduced to

$$f(x)=\mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)} \quad \ldots\ldots\ldots(19).$$

If $f(x)$ be of the form $\frac{\mathrm{f}(x)}{\mathrm{F}(x)}$, and $\mathrm{F}(x)$ be of higher dimensions than $\mathrm{f}(x)$, the expression (18) is, as was proved before, reduced to $\mathfrak{E}\frac{[f(z)]}{x-z}$, which enables us to decompose a rational fraction. But if $\mathrm{F}(x)$ be of lower dimensions than $\mathrm{f}(x)$, the function $\frac{\mathrm{f}(x)}{\mathrm{F}(x)}$ is, by (18), divided into two parts, of which $\mathfrak{E}\frac{[f(z)]}{x-z}$ is the sum of the rational fractions remaining from the division of $\mathrm{f}(x)$ by $\mathrm{F}(x)$, and $\mathfrak{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}$ is the quotient arising from that division.

Let us take, as an example, $f(x) = \dfrac{1+x^4}{x(1+x^2)}$

$$f(x) = \mathfrak{E}\,\frac{(1+z^4)}{[z(1+z^2)]\,(x-z)} + \mathfrak{E}\,\frac{1+z^4}{[z^2]\,(1+z^2)\,(1-zx)}.$$

Now

$$\mathfrak{E}\frac{1+z^4}{[z(1+z^2)](x-z)} = \mathfrak{E}_0\frac{(1+z^4)}{z(1+z^2)(x-z)} + \mathfrak{E}+-^{\frac{1}{2}}1\,\frac{1+z^4}{z(1+z^2)\,(x-z)} + \mathfrak{E}--^{\frac{1}{2}}1\,\frac{1+z^4}{z(1+z^2)\,(x-z)}$$

$$= \frac{1}{x} - \frac{1}{x+-^{\frac{1}{2}}1} - \frac{1}{x--^{\frac{1}{2}}1},$$

and $\mathfrak{E}\,\dfrac{1+z^4}{[z^2]\,(1+z^2)(1-zx)} = \dfrac{d}{dh}\,\dfrac{(1+h^4)}{(1+h^2)\,(1-xh)}$ when $h=0, =x$.

Therefore, finally,

$$\frac{1+x^4}{x(1+x^2)} = x + \frac{1}{x} - \frac{1}{x+-^{\frac{1}{2}}1} - \frac{1}{x--^{\frac{1}{2}}1}$$

$$= x + \frac{1}{x} - \frac{2x}{1+x^2}.$$

The length to which this article has extended, renders it necessary to postpone to a future Number other illustrations of the use of the Residual Calculus.

D. F. G.

II.—TRANSFORMATION OF CERTAIN ANALYTICAL EXPRESSIONS.

We frequently require in Analysis to find the sum of the squares of three quantities of the form

$$ay - bx, \quad bz - cy, \quad cx - az,$$

and a slight artifice enables us to do this very readily.

By adding and subtracting $a^2x^2 + b^2y^2 + c^2z^2$, the sum of the squares becomes

$$(a^2+b^2+c^2)(x^2+y^2+z^2) - (ax+by+cz)^2 \ldots\ldots (1)$$

$$= (a^2+b^2+c^2)(x^2+y^2+z^2)\left\{1 - \frac{(ax+by+cz)^2}{(a^2+b^2+c^2)(x^2+y^2+z^2)}\right\}$$

Now the expression $\dfrac{ax+by+cz}{(a^2+b^2+c^2)^{\frac{1}{2}}(x^2+y^2+z^2)^{\frac{1}{2}}}$ is the cosine of the angle between two lines which make angles with the coordinate

axes, whose cosines are proportional to $a\,b\,c$, $x\,y\,z$ respectively. If this angle be called θ, the expression becomes

$$(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2)\,\sin^2\theta \ldots\ldots (3).$$

We shall give some examples of the use of this formula.

1. In transforming expressions from one set of rectangular coordinates to another, making with the original axes angles whose cosines are $a\,a'a''$, $b\,b'b'$, $c\,c'c''$, we obtain the relations

$$a = \frac{b'c'' - c'b''}{\mathrm{D}},\quad a' = \frac{cb'' - bc''}{\mathrm{D}},\quad a'' = \frac{bc' - cb'}{\mathrm{D}},$$

and we wish to discover the value of D. Squaring these equations, adding and observing that $a^2 + a'^2 + a''^2 = 1$, we find

$$\mathrm{D}^2 = (b'c'' - c'b'')^2 + (cb'' - bc'')^2 + (bc' - cb')^2.$$

By formula (3) this becomes

$$\mathrm{D}^2 = (b^2 + b'^2 + b''^2)\,(c^2 + c'^2 + c''^2)\,\sin^2\theta.$$

Now $\sin\theta = 1$, since the lines are at right angles to each other, the new coordinates being rectangular; also $b^2 + b'^2 + b''^2 = 1$, and $c^2 + c'^2 + c''^2 = 1$, so that $\mathrm{D} = 1$, and

$$a = b'c'' - c'b'',\quad a' = cb'' - bc'',\quad a'' = bc' - cb'.$$

2. Let a material point be acted on by any forces whose resolved parts parallel to rectangular axes are X, Y, Z; let ρ be the radius of curvature of its path at any point xyz, then we know that

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^3 =$$

$$\left\{\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right)^2 + \left(\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right)^2 + \left(\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left(\mathrm{Y}\frac{dx}{dt} - \mathrm{X}\frac{dy}{dt}\right)^2 + \left(\mathrm{Z}\frac{dy}{dt} - \mathrm{Y}\frac{dz}{dt}\right)^2 + \left(\mathrm{X}\frac{dz}{dt} - \mathrm{Z}\frac{dx}{dt}\right)^2\right\}^{\frac{1}{2}}.$$

Therefore, by formula (3)

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^3 = \sqrt{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2}\,\frac{ds}{dt}\cdot \sin(\text{ang. between tang. and resultant}),$$

or if R be the resultant, V the velocity,

$v^2 = \rho\ \mathrm{R}\cos(\text{angle between resultant and rad. of curvature})$.

Calling this angle ϕ, R cos ϕ is the part of the resultant resolved perpendicularly to the path, which is opposed and equal to the centrifugal, which is therefore equal to $\frac{v^2}{\rho}$. $\rho\cos\phi$ is the $\frac{1}{2}$ chord of curvature lying in the direction of the resultant, and therefore

$$v^2 = 2\mathrm{R}\ \tfrac{1}{4}\text{ chord of curvature along resultant,}$$

or the velocity in any curve at any point is equal to that acquired by falling through $\frac{1}{4}$ of the chord of curvature under the influence of the resultant force at the point, considered constant.

3. If p, q, r be the angular velocities of a body round the coordinate axes, the velocities of a point (xyz) parallel to the axes are

$$qz - ry,\ rx - pz,\ py - qx.$$

A point in the line whose equations are

$$\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$$

has no velocity. This line therefore is the axis of instantaneous rotation, and p, q, r are respectively proportional to the cosines of the angles it makes with the coordinate axes.

The velocity of any other point is the square root of the sum of the squares of the velocities parallel to the axes. If this be v we have by (3)

$$v = \sqrt{p^2+q^2+r^2}\,\sqrt{x^2+y^2+z^2}\,.\,\sin(\text{angle between inst. axis and line through } xyz)$$

$$= \sqrt{p^2+q^2+r^2}\,.\,\text{perpendicular on inst. axis.}$$

And $\sqrt{p^2+q^2+r^2}$, being equal to the linear velocity divided by the distance of the point from the instantaneous axis, is the angular velocity about the axis.

4. The expression for the radius of curvature of a curve of double curvature is

$$\rho = \frac{ds^2\,\{(A\,dy - B\,dx)^2 + (C\,dx - A\,dz)^2 + (B\,dz - C\,dy)^2\}^{\frac{1}{2}}}{A^2+B^2+C^2}$$

where

$$A = dy\,d^2z - dz\,d^2y,\ B = dz\,d^2x - dx\,d^2z,\ C = dx\,d^2y - dy\,d^2x,$$

which are proportional to the cosines of the angles which a normal to the osculating plane makes with the coordinate axes. The numerator is, by (3), equal to

$$ds^2\,(A^2+B^2+C^2)^{\frac{1}{2}}\,(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sin^2\theta,$$

when θ is the angle between a normal to the osculating plane and the tangent, and therefore is a right angle, so that $\sin\theta = 1$; and, as $dx^2 + dy^2 + dz^2 = ds^2$, we have

$$\rho = \frac{ds^3}{(A^2+B^2+C^2)^{\frac{1}{2}}}.$$

A. S.

III.—DEMONSTRATIONS OF SOME PROPERTIES OF A TRIANGLE.

THE following properties of a triangle are not perhaps generally known. If we join D, E, F, (Fig 1,) the points in which perpendiculars on the sides from the opposite angles intersect the sides,

the triangles DEC, DBF, AFE are similar to each other, and to the triangle ABC: and the triangle DEF is the triangle of least perimeter, which can be inscribed in the triangle ABC.

We shall first prove that two adjacent sides of the triangle DEF make equal angles with the side of the triangle ABC: that is, that BDF = EDC, DFB = EFA, AEF = DEC.

We have $$BDF = DFC + DCF,$$

as it is the exterior and opposite angle of the triangle DFC. For a similar reason EDC = DBE + DEB.

Now, since the three perpendiculars AD, BE, CF pass through the same point G, and in the quadrilateral BFGD the angles at D and F are together equal to two right angles, the four points B, F, G, D are in the circumference of a circle: and the angles DBG and DFG, or DBE and DFC, being angles in the same segment, are equal to one another. In the same way we see that DCF = DEB, consequently BDF = CDE, and in a similar manner we might prove that BFD = EFA and FEA = DEC.

We shall next prove that EDC = BAC, DEC = ABC, and BFD = BCA. For we have, as before,

$$EDC = DBE + DEB.$$

Now, since ADB and AEB are right angles, they are angles in a semicircle, so that A, B, D, E are points in the circumference of a circle, and DBE and DAE being angles in the same segment of a circle are equal, and also DAB and DEB for the same reason. But BAD + DAE = BAE, so that EDC = BAC; and similarly for the others. Comparing the triangles DEC and ABC, we see that the angle at C is common, the angles CDE = BAC and the angle DEC = ABC, therefore they are equiangular and similar. In the same way DBF and FAE are similar to ABC, and the three triangles DEC, DBF, and FAE, being similar to ABC are similar to one another.

To prove that DEF is the triangle of least perimeter which can be inscribed in ABC, we must avail ourselves of a known property, that, if from two points without a line we draw two straight lines to a point in the line, their sum will be a minimum when the two lines make equal angles with the given line. Now, in this case, having proved that FD and ED make equal angles with BC, we see that, supposing F and E fixed, the sum of FD and DE is the least of all those which can be drawn from F and E to any point in BC. The same holds true of the other points F and E, so that on the whole the perimeter FED is the least of all those which can be inscribed in ABC.

Various other properties may be demonstrated of this triangle of least perimeter. If we call the angle BAD = α, EBC = β, FCA = γ, it is easy to see that FED = 2α, EFD = 2β, FDE = 2γ, and therefore $2(\alpha+\beta+\gamma)$ is equal to two right angles, or $\alpha+\beta+\gamma$ is equal to a right angle. From the construction of the figure it appears that $\alpha = \frac{\pi}{2} - B$, $\beta = \frac{\pi}{2} - C$, $\gamma = \frac{\pi}{2} - A$. Also, since

the angles at D, E, F are bisected by the lines intersecting at G, that point is the centre of the circle inscribed in the triangle of least perimeter. If we draw perpendiculars from the vertices of the triangle ABC to the sides of DEF, it will be seen from the similarity of the triangles that they will divide the angles into the same parts as the perpendiculars on the sides of ABC, but in an inverted position. Now, calling the angles as before α, β, γ, AD makes angles α and β with AF, AE respectively, and therefore the perpendicular on FE makes angles α and β with AE, AF respectively. The position of AD may be determined by expressing the relation between α and β, which may be put under the form $\beta = \phi(\alpha)$. Similarly, the position of BE will be determined by $\gamma = \phi_1(\beta)$, and the position of CF by $\alpha = \phi_2(\gamma)$. But since these lines intersect in one point, one of the three equations must be deducible from the other two, as the combination of two at their point of intersection must coincide with each other. Now the perpendiculars on the sides of the triangle of least perimeter being defined by the same equations, of which one is derivable from the other two, must all pass through one point. This may be clearer if we take the actual relations, which will be found to be

$$\alpha = C - B + \beta$$
$$\beta = A - C + \gamma$$
$$\gamma = B - A + \alpha,$$

the latter of which is clearly derivable from the other two, so that they are not independent, and therefore the lines expressed by them must all pass through one point.

The perimeter of the triangle DEF may be easily found. For, from the similarity of the triangles ABC, DEF, we have

$$FD : BD = AC :: AB = b : c, \text{ or } FD = \frac{b}{c} BD,$$

$$\text{but } BD = c \cos B, \text{ so that } FD = b \cos B.$$

Similarly, $DE = c \cos C$, and $FE = a \cos A$, so that if p be the perimeter of DEF,

$$p = a \cos A + b \cos B + c \cos C.$$

We may likewise obtain an expression for the perimeter involving only the sides of the triangle ABC. The area of the triangle DEF

$$= \tfrac{1}{2} FD \,.\, DE \sin 2\gamma = \tfrac{1}{2} bc \cos B \cos C \sin 2A.$$

Since $$\gamma = \frac{\pi}{2} - A \text{ and } 2\gamma = \pi - 2A.$$

Now if r be the radius of the circle inscribed in DEF, its area $= \frac{1}{2} pr$.

But $$r = GD \sin \gamma = GD \cos A = CD \tan \alpha \cos A$$
$$= b \cos C \cos A \tan \alpha = b \frac{\cos A \cos B \cos C}{\sin B}.$$

Therefore

$$\tfrac{1}{2}bp\,\frac{\cos A\cos B\cos C}{\sin B} = \tfrac{1}{2}bc\cos B\cos C\sin 2A$$
$$= bc\cos A\cos B\cos C\sin A,$$
$$\text{or } p = 2c\sin A\sin B.$$

Or, substituting for sin A and sin B, their values in terms of the sides of ABC,

$$p = 2c\,\frac{2}{bc}\cdot\frac{2}{ac}\,\{S(S-a)(S-b)(S-c)\}$$

$$\text{or } p = \frac{8}{abc}\,\{S(S-a)(S-b)(S-c)\}.$$

If R′ be the radius of the circle described round ABC, we have, by taking its value in terms of the sides of the triangle,

$$p = \frac{abc}{2R'^2}.$$

Also, if R be the radius of the circle inscribed in ABC, we know that generally,

$$2RR' = \frac{abc}{a+b+c} = \frac{abc}{P},$$

if P be the perimeter of ABC. Substituting the value of R′², from this we have

$$p = 2\,\frac{P^2R^2}{abc}.$$

And therefore if K be the area of ABC,

$$p = \frac{8K^2}{abc}.$$

The area of DEF is very easily found in terms of the area of ABC, for we found before that it was equal to

$$\tfrac{1}{2}bc\sin 2A\cos B\cos C = bc\sin A\cos A\cos B\cos C$$
$$= 2K\cos A\cos B\cos C = k \text{ suppose.}$$

If a', b', c' be the sides of DEF,

$$a' = a\cos A,\quad b' = b\cos B,\quad c' = c\cos C,$$

and therefore substituting for cos A cos B cos C,

$$k = 2K\,\frac{a'b'c'}{abc}.$$

But we have $abc = 2PRR' = 4KR'$,

$$\text{and therefore } k = \frac{a'b'c'}{2R}.$$

It will be seen at once that the areas of the triangles FAE, DBF, DCE, are respectively

$$K\cos^2 A,\quad K\cos^2 B,\quad K\cos^2 C;$$

as the sum of these together with DEF make up ABC, we have, dividing by K,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\frac{a'b'c'}{abc} = 1.$$

A singular relation exists between the radii of the circles described round ABC and DEF. Let them be R′ and r' respectively. Then,

$$a = 2R' \sin A, \quad a' = 2r' \sin 2\gamma = 2r' \sin 2A.$$

Therefore $a' = a \cos A = 4r' \sin A \cos A.$

Whence $R' = 2r'.$

We found for r the radius of the circle inscribed in DEF.

$$r = \frac{b}{\sin B} \cos A \cos B \cos C,$$

$$\text{and as } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R';$$

$$\text{therefore } r = 2R' \cos A \cos B \cos C = R' \frac{k}{K}.$$

Also, since $r = \frac{2k}{p} = \frac{4K}{p} \cos A \cos B \cos C,$

$$R' = \frac{2K}{p}, \text{ and } r' = \frac{K}{p}.$$

Since the sides of any one of the triangles round DEF are equal to the sides of the original triangle multiplied by the cosine of the corresponding angle, all similarly situate lines in the triangles, and therefore the radii of the circumscribing circles, will have the same ratio.

Now the circle described round FAE passes through G, as A, F, E, G, are in the circumference of the same circle: and as AEG, AFG are right angles, AG must be the diameter of the circle, therefore $AG = 2R' \cos A.$

Similarly, $BG = 2R' \cos B,$ $CG = 2R' \cos C.$

We find also,

$GD = 2R' \cos B \cos C,$ $GE = 2R' \cos A \cos C,$ $GF = 2R' \cos A \cos B.$

To these geometrical properties we may add the following optical one.

If the interior sides of the triangle be reflecting surfaces, and a ray of light be incident on one of the sides in a direction parallel to one of the sides of the inscribed triangle as to FD, it will, after two reflexions from each side, return to its original direction so as to continue the same course *ad infinitum*. This property depends of course on the equality of the angles of incidence and reflection, so that the ray being incident parallel to one of the sides of FDE will continue its course successively parallel to the other sides. Let *fd* be the incident ray which being incident parallel to FD will be reflected at *d* parallel to DE, and falling on AC at *e* is reflected

parallel to EF meeting AB in f'. Proceeding from that point in the same manner, it will after reflection again from BC and AC meet AB again in some point f'': our object is to show that f'' coincides with f. For if it does so, it is clear from the course of the ray being parallel to the sides of EFD that it will go over the same track again, and so on.

Since the triangles Bdf, ABC are similar, we have BC.Bd = AB Bf. For the like reason we have

$$\begin{aligned} \mathrm{BC}\,.\,\mathrm{C}d &= \mathrm{AC}\,.\,\mathrm{C}e \\ \mathrm{AB}\,.\,\mathrm{A}f' &= \mathrm{AC}\,.\,\mathrm{A}e \\ \mathrm{AB}\,.\,\mathrm{B}f'' &= \mathrm{BC}\,.\,\mathrm{B}d' \\ \mathrm{AC}\,.\,\mathrm{C}e' &= \mathrm{BC}\,.\,\mathrm{C}d' \\ \mathrm{AC}\,.\,\mathrm{A}e' &= \mathrm{AB}\,.\,\mathrm{A}f''. \end{aligned}$$

Adding these together, and observing that we have

$$\mathrm{B}d + \mathrm{C}d = \mathrm{BC},\quad \mathrm{C}e + \mathrm{A}e = \mathrm{AC},\quad \mathrm{A}f' + \mathrm{B}f' = \mathrm{AB}$$
$$\mathrm{A}e' + \mathrm{C}e' = \mathrm{AC},\quad \mathrm{B}d' + \mathrm{C}d' = \mathrm{BC},$$

we have

$$\mathrm{BC}^2 + \mathrm{AB}^2 + \mathrm{AC}^2 = \mathrm{AC}^2 + \mathrm{BC}^2 + \mathrm{AB}\,.\,\mathrm{B}f + \mathrm{AB}\,.\,\mathrm{A}f'',$$
$$\text{whence}\quad \mathrm{AB}^2 = \mathrm{AB}\,(\mathrm{B}f + \mathrm{A}f''),$$
$$\text{or}\quad \mathrm{AB} = \mathrm{B}f + \mathrm{A}f''.$$

Therefore $\mathrm{A}f = \mathrm{A}f''$ and f' and f'' coincide, and the proposition is manifest.

This and several of the other properties of the triangle proved here are due to Professor Wallace of Edinburgh.

V.

IV.—ON THE SOLUTION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS.*

Integration of the Partial Differential Equation

$$\mathrm{X}^a\mathrm{Y}^b\mathrm{Z}^c\left(\frac{d\mathrm{Z}}{d\mathrm{X}}\right)^m\left(\frac{d\mathrm{Z}}{d\mathrm{Y}}\right)^n - \mathrm{A} = 0,$$

a, b, c, m, n being any quantities whatever.

This equation may be put under the form

$$\left\{\mathrm{X}^{\frac{a}{m}}\,\mathrm{Z}^{\frac{c}{m+n}}\,\frac{d\mathrm{Z}}{d\mathrm{X}}\right\}^m \cdot \left\{\mathrm{Y}^{\frac{b}{n}}\,\mathrm{Z}^{\frac{c}{m+n}}\,\frac{d\mathrm{Z}}{d\mathrm{Y}}\right\}^n - \mathrm{A} = 0.$$

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* From a Correspondent.

Let $x = \int X^{-\frac{a}{m}} dX = \frac{m}{m-a} X^{\frac{m-a}{m}}$

$y = \int Y^{-\frac{b}{n}} dY = \frac{n}{n-b} Y^{\frac{n-b}{n}}$, $z = \int Z^{\frac{c}{m+n}} dZ = \frac{m+n}{m+n+c} Z^{\frac{m+n+c}{m+n}}$,

and the equation becomes

$$\left(\frac{dz}{dx}\right)^m \left(\frac{dz}{dy}\right)^n - A = 0 \ldots\ldots\ldots (1),$$

which is easily integrable.

If $m = a$, then $x = \log_e X$; if $n = b$, then $y = \log_e Y$; and if $m + n + c = 0$, then $Z = \log_e Z$.

But if $m + n = 0$, and c does not $= 0$, then (and in this case only) the above substitution for Z will not answer, and equation (1) must stand thus,

$$\left(\frac{dZ}{dx}\right)^m \left(\frac{dZ}{dy}\right)^n Z^c - A = 0 \ldots\ldots\ldots (2).$$

But by considering x as a function of the independent variables Z and y, we may transform this equation so as to be easily integrated.

$$\text{For } dx = \frac{dx}{dZ} dZ + \frac{dx}{dy} dy;$$

$$\therefore dZ = \frac{1}{\frac{dZ}{dx}} dx - \frac{\frac{dx}{dy}}{\frac{dZ}{dx}} dy;$$

$$\text{and hence } \frac{dZ}{dx} = 1 \div \frac{dx}{dZ}, \text{ and } \frac{dZ}{dy} = -\frac{dx}{dy} \div \frac{dx}{dZ};$$

and equation (2) becomes by substitution (since $m + n = 0$)

$$\left(\frac{dx}{dy}\right)^n Z^c - A(-1)^n = 0,$$

the integral of which is

$$x = -y Z^{-\frac{c}{n}} A^{\frac{1}{n}} + F(Z).$$

The same process may be applied when there is any number of independent variables.

Ex. 1. $\frac{dz}{dy} = \frac{dz^2}{dx^2} xy^2 z^3$.

This may be written $\frac{z^3}{y^2} \frac{dz}{dy} = \left(x^{\frac{1}{2}} z^3 \frac{dz}{dx}\right)^2$:

put $\frac{y^3}{3} = y'$, $2x^{\frac{1}{2}} = x'$, and $\frac{z^4}{4} = z'$, and we have

$$\frac{dz'}{dy'} = \frac{dz'^2}{dx'^2}.$$

To integrate this we proceed as follows:

$$dz' = \frac{dz'}{dx'}\,dx' + \frac{dz'}{dy'}\,dy' = p\,dx' + q\,dy' \text{ suppose;}$$

$$\therefore z' = px' + qy' - \int(x'dp + y'dq) = px' + p^2y' - \int(x' + 2py')\,dp;$$

$$\left.\begin{aligned} \therefore\ x' + 2py' &= \mathrm{F}'(p), \\ \text{and}\quad z' &= px' + p^2y' - \mathrm{F}(p), \end{aligned}\right\}$$

from which p must be eliminated.

If we restore the original variables, these equations are

$$\left.\begin{aligned} 2\sqrt{x} + \tfrac{2}{3}y^3p &= \mathrm{F}'(p), \\ \tfrac{1}{4}z^4 &= 2\sqrt{x}p + \tfrac{1}{3}y^3p^2 - \mathrm{F}(p). \end{aligned}\right\}$$

Ex. 2. $\dfrac{dz}{dy}xz = \dfrac{dz}{dx}$:

this comes under the case where $m + n = 0$, and c does not $= 0$.

Put $\dfrac{x^2}{2} = x'$, and the equation becomes

$$\frac{dz}{dy}z = \frac{dz}{dx'},$$

$$dx' = \frac{dx'}{dz}\,dz + \frac{dx'}{dy}\,dy;$$

$$\therefore\ dz = \left(1 \div \frac{dx'}{dz}\right)dx' - \left(\frac{dx'}{dy} \div \frac{dx'}{dz}\right)dy;$$

$$\therefore\ \frac{dz}{dx'} = 1 \div \frac{dx'}{dz}, \text{ and } \frac{dz}{dy} = -\frac{dx'}{dy} \div \frac{dx'}{dz},$$

and our equation becomes

$$\frac{dx'}{dy}z = -1;$$

$$\therefore\ x' = -\frac{y}{z} + \mathrm{F}(z),$$

$$\text{or}\quad \frac{x^2}{2} + \frac{y}{z} = \mathrm{F}(z).$$

G. C.

V.—ON THE VARIATION OF THE LONGITUDE OF PERIHELION IN THE PLANETARY THEORY.

THE following is a short method of finding the expression for $\frac{d\omega}{dt}$. The usual notation is employed.

We have the equation

$$\frac{1}{r} = \frac{\mu}{h^2}\{1 + e\cos(\theta - \omega)\} \ldots\ldots\ldots(1);$$

$$\text{and since} \quad 1 - e^2 = \frac{h^2}{\mu a},$$

e is a function of a and h, therefore

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} + \frac{dr}{da}\frac{da}{dt} + \frac{dr}{dh}\frac{dh}{dt} + \frac{dr}{d\omega}\frac{d\omega}{dt},$$

$$\text{but} \quad \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt};$$

therefore, observing that $\frac{dr}{d\omega} = -\frac{dr}{d\theta}$,

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = \frac{dr}{da}\frac{da}{dt} + \frac{dr}{dh}\frac{dh}{dt}.$$

Now $\frac{da}{dt} = -\frac{2a^2}{\mu}\frac{d(\mathrm{R})}{dt}$, and $\frac{dh}{dt} = -\frac{d\mathrm{R}}{d\theta}$; therefore

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = -\frac{2a^2}{\mu}\frac{dr}{da}\frac{d(\mathrm{R})}{dt} - \frac{dr}{dh}\frac{d\mathrm{R}}{d\theta},$$

$$\text{and} \quad \frac{d(\mathrm{R})}{dt} = \frac{d\mathrm{R}}{dr}\frac{dr}{dt} + \frac{d\mathrm{R}}{d\theta}\frac{d\theta}{dt}; \text{ therefore}$$

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = -\frac{2a^2}{\mu}\frac{dr}{da}\frac{dr}{dt}\frac{d\mathrm{R}}{dr} - \left(\frac{2a^2}{\mu}\frac{dr}{da}\frac{d\theta}{dt} + \frac{dr}{dh}\right)\frac{d\mathrm{R}}{d\theta}$$

$$= -\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da}\frac{dr}{d\theta}\frac{d\mathrm{R}}{dr} - \left(\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da} + \frac{dr}{dh}\right)\frac{d\mathrm{R}}{d\theta} \ldots(2),$$

$$\text{putting} \quad \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}, \quad \text{and} \quad \frac{d\theta}{dt} = \frac{h}{r^2}.$$

Now, differentiating (1) relatively to a, we have

$$-\frac{1}{r^2}\frac{dr}{da} = \frac{\mu}{h^2}\frac{de}{da}\cos(\theta - \omega) \ldots\ldots\ldots(3);$$

and since $1 - e^2 = \frac{h^2}{\mu a}$,

$$-2e\frac{de}{da} = -\frac{h^2}{\mu a^2}, \quad \text{and} \quad -2e\frac{de}{dh} = \frac{2h}{\mu a},$$

whence $\frac{de}{da} = -\frac{h}{2a}\frac{de}{dh}$, and (3) becomes

$$-\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da} = -a\frac{de}{dh}\cos(\theta-\omega) \quad \ldots\ldots\ldots\ldots (4).$$

Also, $r = a(1 - e\cos u)$,

$$u = nt + \varepsilon - \omega + e\sin u;$$

therefore $\frac{d(r)}{dh} = -a\frac{de}{dh}\cos u + ae\sin u\frac{du}{dh}$,

and $\frac{du}{dh} = \frac{de}{dh}\sin u + e\cos u\frac{du}{dh}$,

hence $\frac{du}{dh} = \frac{de}{dh}\cdot\frac{\sin u}{1 - e\cos u}$,

$$\frac{d(r)}{dh} = -a\frac{de}{dh}\left(\cos u - \frac{e\sin^2 u}{1 - e\cos u}\right)$$

$$= -a\frac{de}{dh}\frac{\cos u - e}{1 - e\cos u}$$

$$= -a\frac{de}{dh}\cos(\theta - \omega);$$

hence, by (4),

$$-\frac{2a^2}{\mu}\frac{h}{r^2}\frac{dr}{da} = \frac{d(r)}{dh}.$$

Substituting in (2)

$$\frac{dr}{d\theta}\frac{d\omega}{dt} = \frac{dr}{d\theta}\frac{d\mathrm{R}}{dr}\frac{d(r)}{dh} + \frac{d\mathrm{R}}{d\theta}\left(\frac{d(r)}{dh} - \frac{dr}{dh}\right):$$

but $\frac{d(r)}{dh} = \frac{dr}{dh} + \frac{dr}{d\theta}\frac{d(\theta)}{dh}$;

therefore $\frac{d\omega}{dt} = \frac{d\mathrm{R}}{dr}\frac{d(r)}{dh} + \frac{d\mathrm{R}}{d\theta}\frac{d(\theta)}{dh}$

$$= \frac{d\mathrm{R}}{dh}$$

$$= \frac{d\mathrm{R}}{de}\frac{de}{dh}$$

$$= -\frac{na\sqrt{1-e^2}}{\mu e}\frac{d\mathrm{R}}{de}.$$

M. O. B.

VI.—ON THE OSCILLATIONS OF A FLOATING BODY.

In all the treatises on Hydrostatics in use in the University, the vertical and angular oscillations of floating bodies are treated of in separate propositions, on the supposition that either of the two kinds of oscillations can take place without the other. It is shewn in Webster's Theory of Fluids, Art. 69, that unless the vertical line through the centres of gravity of the body and of the fluid displaced by it in the position of equilibrium passes through the centre of gravity of the plane of floatation, that is, unless the body be symmetrical with respect to this line, there cannot be angular oscillations unattended with vertical, and it might as readily be shewn that there cannot be vertical oscillations unattended with angular. It is not difficult, as will be seen, to determine the simultaneous oscillations of both kinds, provided that the angular oscillations are confined to one plane, namely, that passing through the three centres of gravity above mentioned, which may evidently be the case when the body is symmetrical with respect to this plane.

In Fig. 2, AB is the projection of the plane of floatation in equilibrium, ab the plane of floatation at any time, C the centre of gravity of the plane AB, $a'b'$ parallel to ab through C, G the centre of gravity of the body. Let x be the vertical distance of the centre of gravity of the body below its place of rest at the time t, θ the angular deviation from the position of equilibrium. Then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{M' - M}{M}g,$$

$$\frac{d^2\theta}{dt^2} = \frac{M'y}{Mk^2}g,$$

M being the mass of the body, M′ that of the fluid displaced, y the horizontal distance of the centre of gravity of the fluid displaced from that of the body, Mk^2 the moment of inertia of the body about its centre of gravity.

We must express approximately M′ and M′y in terms of x and θ. Let Gh be perpendicular to AB, Ck to ab, and let K be the area of the section AB, a the distance Ch, α the angle CGh, ρ the density of the fluid.

Then $M' = \rho \times$ vol. $aDb = \rho \times$ (vol. $a'Db' +$ vol. $aa'b'b$), and vol. $a'Db' =$ vol. ADB, (Webster's Theory of Fluids, Art. 68)

therefore $\rho \times$ vol. $a'Db' = M$.

The vol. $aa'b'b = K \times Ck$ nearly:

now x is the sum of the distances through which the centre of gravity is depressed both by the angular and vertical motions. In the figure the centre of gravity is raised by the angular motion, therefore

$$x = Ck - CG\{\cos\alpha - \cos(\alpha + \theta)\}$$
$$= Ck - \theta \,.\, CG \sin\alpha, \text{ nearly:}$$

therefore $Ck = x + a\theta$,

$$M' = M + \rho K (x + a\theta),$$

$$\frac{d^2x}{dt^2} = -\frac{\rho K g}{M}(x + a\theta) \ldots\ldots\ldots(1).$$

Again $M'y$ = moment of aDb = moment $a'Db'$ + moment $a'a'\ bb$. As is proved in all treatises on Hydrostatics in determining the equilibrium of a floating body,

$$\text{moment } a'Db' = (\rho I - Mb)\,\theta,$$

I being the moment of inertia of the plane AB about an axis through C perpendicular to the plane ADB, and b the distance of the centre of gravity of the fluid displaced *below* that of the body.

The moment of $aa'\ b'b = \rho K \times Ck \times CG \sin(a + \theta)$ nearly

$$= \rho K (x + a\theta)\, a \text{ nearly},$$

$$\text{therefore } M'y = \rho K a x + \{\rho (I + Ka^2) - Mb\}\,\theta,$$

$$\text{hence } \frac{d^2\theta}{dt^2} = -\frac{\rho K a g}{Mk^2} x - \frac{\{\rho (I + Ka^2) - Mb\}\, g}{Mk^2} \cdot \theta \ldots\ldots\ldots(2),$$

(1) and (2) are simultaneous equations which may be solved in the usual way, and thus x and θ expressed in terms of t. It will be convenient to assume $\frac{\rho K g}{M} = m^2$, and $\frac{\rho (I + Ka^2) - Mb}{\rho K a} = c$, then they become

$$\frac{d^2x}{dt^2} + m^2 (x + a\theta) = 0,$$

$$\frac{d^2\theta}{dt^2} + \frac{m^2 a}{k^2}(x + c\theta) = 0.$$

S. S. G.

VII.—NEW ANALYTICAL METHODS OF SOLVING CERTAIN GEOMETRICAL PROBLEMS.

THE problems which will form the subject of this Article, are some in which it is required to show, that the points of intersection of certain lines, straight or curved, described according to given conditions, lie in some other assigned line. The methods of solution will be best explained in applying them to particular propositions.

The first that we shall prove, is one from the Senate-House Problems of the present year: namely,

If on any three chords drawn through the same point in the circumference of a circle, as diameters three circles will be described, the points of intersection of these circles, two and two, lie in the same straight line.

Let a be the radius of the principal circle, α, β, γ the angles which the chords make with the diameter which meets them; let the point through which the chords are drawn be taken for the origin, and the diameter of the principal circle for the axis of x.

The length of the chord whose inclination is α, is $2a \cos \alpha$, and the coordinates of its middle point are $a (\cos \alpha)^2$, $a \cos \alpha \sin \alpha$: hence the equation to the circle described upon it, is

$$\{x - a (\cos \alpha)^2\}^2 + \{y - a \cos \alpha \sin \alpha\}^2 = a^2 (\cos \alpha)^2,$$

$$\text{or } x^2 + y^2 - 2ax (\cos \alpha)^2 - 2ay \cos \alpha \sin \alpha = 0 \ldots\ldots (1).$$

Let $\tan \alpha$, $\tan \beta$, $\tan \gamma = \lambda$, μ, ν respectively: then (1) becomes

$$\rho^2 (1 + \lambda^2) - 2ax - 2ay\lambda = 0 \ldots\ldots\ldots (2),$$

putting $x^2 + y^2 = \rho^2$ for abbreviation. Similarly the equation to the second circle is

$$\rho^2 (1 + \mu^2) - 2ax - 2ay\mu = 0 \ldots\ldots\ldots (3).$$

Subtracting (2) from (1), and dividing by $\lambda - \mu$, we have

$$\rho^2 (\lambda + \mu) - 2ay = 0 \ldots\ldots\ldots (4).$$

Subtracting (1) multiplied by μ, from (2) multiplied by λ, and dividing by $\lambda - \mu$,

$$\rho^2 (1 - \lambda\mu) - 2ax = 0 \ldots\ldots\ldots (5).$$

Since these last two equations are obtained by combining (2) and (3), they will be satisfied by the coordinates of the points of intersection of the circles represented by those equations. The next object is to find a relation between $\lambda + \mu$, $\lambda\mu$, and symmetrical functions of all the three quantities λ, μ, ν. For this purpose, let

$$\xi^3 - p\xi^2 + q\xi - r = 0$$

be the equation whose roots are λ, μ, ν: then

$$\lambda^3 - p\lambda^2 + q\lambda - r = 0,$$

$$\mu^3 - p\mu^2 + q\mu - r = 0.$$

Performing on these equations the same processes as before on (2) and (3), we obtain

$$\lambda^2 + \lambda\mu + \mu^2 - p (\lambda + \mu) + q = 0,$$

$$\text{or } (\lambda + \mu)^2 - \lambda\mu - p(\lambda + \mu) + q = 0,$$

$$\text{and } \lambda\mu (\lambda + \mu) - p\lambda\mu + r = 0.$$

Substituting in these two last equations the values of $\lambda + \mu$ and $\lambda\mu$ derived from (4) and (5), we have

$$\frac{4a^2y^2}{\rho^4} - 1 + \frac{2ax}{\rho^2} - p\frac{2ay}{\rho^2} + q = 0,$$

$$\text{and } \left(\frac{2ay}{\rho^2} - p\right)\left(1 - \frac{2ax}{\rho^2}\right) + r = 0:$$

$$\text{or } \frac{4a^2y^2}{\rho^4} + \frac{2a}{\rho^2}(x - py) + q - 1 = 0 \ldots\ldots\ldots (6)$$

$$\text{and } -\frac{4a^2xy}{\rho^4} + \frac{2a}{\rho^2}(px + y) - p + r = 0 \ldots\ldots\ldots (7).$$

These two equations, on account of the manner in which they were obtained, must both be satisfied by the coordinates of the points of intersection of the circles represented by (2) and (3). But, since they involve λ, μ, ν symmetrically, they must equally be satisfied by the coordinates of the points of intersection of the two other pairs of circles. And since their number is the same as that of the unknown quantities, x and y, we may, by combining them in various ways, obtain every other equation that is satisfied by those coordinates, and among them, the required relation. This is done by multiplying (6) by x and (7) by y and adding; whence, observing that $x^2 + y^2 = \rho^2$, we have

$$(q - 1)\,x - (p - r)\,y + 2a = 0,$$

the equation to a straight line. Substituting for p, q, r their values in terms of the roots, replacing λ, μ, ν by tan α, tan β, tan γ, and reducing, we find

$$x \cos(\alpha+\beta+\gamma) + y \sin(\alpha+\beta+\gamma) = 2a \cos\alpha \cos\beta, \cos\gamma.$$

The same method applies to the following problem, of which several solutions were given in the Philosophical Magazine in the last two years.

If three tangents be drawn to a parabola, the circle described through their points of intersection will also pass through the focus.

Let λ, μ, ν be the tangents of the angles which the tangents make with the axis, then their equations are

$$y = \lambda x + \frac{m}{\lambda}$$

$$y = \mu x + \frac{m}{\mu}$$

$$y = \nu x + \frac{m}{\nu}.$$

Subtracting the 2nd from the 1st, and dividing by $\lambda - \mu$,

$$0 = x - \frac{m}{\lambda\mu}.$$

Multiplying the 1st and 2nd by λ and μ respectively, subtracting, and dividing by $\lambda - \mu$,

$$y = (\lambda + \mu)\,x.$$

Now if $$\xi^3 - p\xi^2 + q\xi - r = 0$$

be the equation whose roots are λ, μ, ν, we have, as in the former problem,

$$(\lambda + \mu)^2 - \lambda\mu - p(\lambda + \mu) + q = 0,$$

$$\text{and } (\lambda + \mu - p)\lambda\mu + r = 0.$$

Substituting the values of $\lambda + \mu$ and $\lambda\mu$ in terms of x and y,

$$y^2 - mx - pxy + qx^2 = 0 \ldots\ldots\ldots(1).$$

$$my - mpx + rx^2 = 0 \ldots\ldots\ldots(2).$$

We must combine these equations so as to obtain the equation to a circle. For this purpose, first multiply (2) by y and (1) by m, and subtract; then, after dividing by x,

$$rxy - mqx + m^2 = 0 \ldots\ldots\ldots (3).$$

Next, multiply (1) by r and (3) by p, and add, in order to eliminate the term xy, then

$$ry^2 - rqx^2 - m(r + pq)x + m^2p = 0 \ldots\ldots\ldots (4).$$

Finally, multiply (2) by $1 - q$, and add it to (4), therefore

$$r(y^2 + x^2) + m(1 - q)y - m(p + r)x + m^2p = 0,$$

which is the equation to a circle. It is easy to see that it is satisfied by the values $y = 0$, $x = m$, therefore it passes through the focus.

If α, β, γ be the angles which the tangents make with the axis, and for p, q, r be substituted their values in terms of tan α, tan β, tan γ, after putting the equation under the form

$$r(y^2 + x^2 - m^2) + m(1 - q)y - m(p + r)(x - m) = 0,$$

it becomes

$$\sin\alpha \sin\beta \sin\gamma (y^2 + x^2 - m^2) + m \cos(\alpha + \beta + \gamma)y - m \sin(\alpha + \beta + \gamma)(x - m) = 0.$$

The succeeding problems are of a somewhat different nature, and therefore the same method is not applicable to them, but that which we shall employ, is, we believe, in some degree new. The first is from the Senate-House papers of this year.

HKL, PQR are two triangles, prove that if the straight lines HP, KQ, LR meet in one point, the intersections of KL, QR; LH, RP; HK, PQ lie in a straight line. (Fig. 3.)

Let O be the point in which HP, KQ, LR meet, and take it for the origin of coordinates; let ab, $a'b'$, $a''b''$, be rectangular coordinates of H, K, L, and AB, A′B′, A″B″ of P, Q, R.

Then, since O, H, P are in a straight line,

$$\frac{A}{a} = \frac{B}{b}.$$

Let this ratio be α, and similarly let

$$\frac{A'}{a'} = \frac{B'}{b'} = \alpha', \text{ and } \frac{A''}{a''} = \frac{B''}{b''} = \alpha''.$$

The equation to HK is

$$(b - b')x - (a - a')y = a'b - ab' \ldots\ldots\ldots (1),$$

and that to PQ,

$$(B - B')x - (A - A')y = A'B - AB',$$

or $$(ab - a'b')x - (a\alpha - a'\alpha')y = a\alpha'(a'b - ab') \ldots\ldots\ldots (2).$$

Let $$nx - my = 1 \ldots\ldots\ldots (3)$$

be the equation to a line passing through the intersection of (1)

and (2); then, eliminating x and y by cross multiplication, we obtain the following equation between m and n,

$$\{aa'(b-b')-(ab-a'b')\}m-\{aa'(a-a')-(aa-a'a')\}n + a-a'=0 \ldots\ldots\ldots (4).$$

The same line will also pass through the intersections of the other pairs of lines, if the same values of m and n satisfy these two other equations

$$\{a'a''(b'-b'')-(a'b'-a''b'')\}m-\{a'a''(a'-a'')-(a'a'-a''a'')\}n + a'-a''=0 \ldots\ldots\ldots (5),$$

$$\{a''a(b''-b)-(a''b''-ab)\}m-\{a''a(a''-a)-(a''a''-aa)\}n + a''-a=0 \ldots\ldots\ldots (6);$$

but this is the case, for whether we add (4), (5), (6) as they stand, or after multiplying them by a, a', a'' respectively, in both ways we get the same relation, whence it follows that any one of the equations is a consequence of the other two. To find the value of m, multiply by $a''a''$, aa, $a'a'$, and add, then n disappears, and we find

$$m=[(a''-a')aa+(a-a'')a'a'+(a'-a)a''a''],$$

divided by

$$[aa'a''\{a(b'-b'')+a'(b''-b)+a''(b-b')\}-(a'b'-a''b'')aa - (a''b''-ab)a'a'-(ab-a'b')a''a''].$$

In like manner,

$$n=[(a''-a')ab+(a-a'')a'b'+(a'-a)a''b''],$$

divided by

$$[aa'a''\{a(b'-b'')+a'(b''-b)+a''(b-b')\}-(a'b'-a''b'')aa - (a''b''-ab)a'a'-(ab-a'b')a''a''].$$

If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in a straight line.

Let $$(x-\alpha_1)^2+(y-\beta_1)^2=\gamma_1^2$$

be the equation to one of the circles. The equation to a tangent line making an angle θ with the axis of x, is

$$(x-\alpha_1)\sin\theta-(y-\beta_1)\cos\theta=\gamma_1 \ldots\ldots\ldots (1).$$

If the same line touch another circle whose equation is

$$(x-\alpha_2)^2+(y-\beta_2)^2=\gamma_2^2,$$

we must have

$$\alpha_1\sin\theta-\beta_1\cos\theta+\gamma_1=\alpha_2\sin\theta-\beta_2\cos\theta+\gamma_2,$$

$$\text{or } (\alpha_1-\alpha_2)\sin\theta-(\beta_1-\beta_2)\cos\theta+\gamma_1-\gamma_2=0 \ldots\ldots\ldots (2),$$

which will give two values of θ.

To find the point of intersection of the two tangents, we observe that if x and y in (1) belong to that point, that equation must be satisfied by both the values of θ given by (2). Therefore (1) and (2) must be identical, whence

$$\frac{x-\alpha_1}{\gamma_1} = \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}, \quad \frac{y-\beta_1}{\gamma_1} = \frac{\beta_1-\beta_2}{\gamma_1-\gamma_2},$$

$$\text{and } x = \frac{\alpha_2\gamma_1-\alpha_1\gamma_2}{\gamma_1-\gamma_2}, \quad y = \frac{\beta_2\gamma_1-\beta_1\gamma_2}{\gamma_1-\gamma_2}.$$

Let $$mx + ny = 1$$

be the equation to a line passing through this point, then

$$(\alpha_2\gamma_1-\alpha_1\gamma_2)m+(\beta_2\gamma_1-\beta_1\gamma_2)n = \gamma_1-\gamma_2 \ldots\ldots\ldots (3),$$

and the same line will pass through the intersections of the other pairs of tangents, if values of m and n exist which satisfy also the two equations

$$(\alpha_3\gamma_2-\alpha_2\gamma_3)m+(\beta_3\gamma_2-\beta_2\gamma_1)n = \gamma_2-\gamma_1 \ldots\ldots\ldots (4),$$

$$(\alpha_1\gamma_3-\alpha_3\gamma_1)m+(\beta_1\gamma_3-\beta_3\gamma_1)n = \gamma_3-\gamma_1 \ldots\ldots\ldots (5);$$

but this is true, as is seen by multiplying (3), (4), (5) by γ_3, γ_1, γ_2, and adding, when the whole will vanish. The value of m is found by multiplying by β_3, β_1, β_2, and adding; and that of n by multiplying by α_3, α_1, α_2. Thus we find the equation to be

$$\begin{aligned}&\{\beta_1(\gamma_2-\gamma_3)+\beta_2(\gamma_3-\gamma_1)+\beta_3(\gamma_1-\gamma_2)\}x\\ &-\{\alpha_1(\gamma_2-\gamma_3)+\alpha_2(\gamma_3-\gamma_1)+\alpha_3(\gamma_1-\gamma_2)\}y\\ &=(\alpha_2\beta_3-\alpha_3\beta_2)\gamma_1+(\alpha_3\beta_1-\alpha_1\beta_3)\gamma_2+(\alpha_1\beta_2-\alpha_2\beta_1).\end{aligned}$$

S. S. G.

VIII.—ON THE INTEGRATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

In the present article we shall apply to Simultaneous Linear Differential Equations the principles, the applications of which we have developed in the preceding Numbers of this Journal. The usual method for solving these equations was first given by D'Alembert, and has received but little improvement since his time, although it is so long and tedious, that some change was highly desirable. The process which we shall give here is at once simple and direct, and shows the advantage of recurring frequently to the principles on which our calculations are founded. The theory of the method is sufficiently simple. Since we have shown that the symbols of differentiation are subject to the same laws of combination as those of number, they may be always treated in the same manner if the coefficients be all constants, which is the only case we shall consider. We have therefore only to separate the symbol of differentiation from its subject, and then proceed to eliminate one of the variables between the given equations, exactly as if the symbol of differentiation were an ordinary coefficient. Thus the difficulty of

elimination becomes reduced to that between ordinary and algebraical equations; and all the facilities for effecting it which have been invented for these in particular cases, may be used for differential equations. When the equations involve various orders of differentials of a variable as well as multiples of it, the symbols of differentiation and of number must be grouped together as one coefficient for the variable. Thus, if we had

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\frac{d^2y}{dt^2} + c\frac{dy}{dt} + mx + my = \mathrm{T},$$

$$\frac{d^2y}{dt^2} + a'\frac{dy}{dt} + b'\frac{d^2x}{dt^2} + c'\frac{dx}{dt} + m'y + n'z = \mathrm{T}';$$

they are to be written

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + m\right)x + \left(b\frac{d^2}{dt} + c\frac{d}{dt} + n\right)y = \mathrm{T},$$

$$\left(\frac{d^2}{dt^2} + a'\frac{d}{dt} + m'\right)y + \left(b\frac{d^2}{dt^2} + c'\frac{d}{dt} + n'\right)x = \mathrm{T}',$$

and then treated as if each symbol of operation were a factor.

We shall proceed to give some examples; and first let us take

$$\frac{dx}{dt} + ay = 0,$$

$$\frac{dy}{dt} + bx = 0.$$

We have to separate $\frac{d}{dt}$ from the variable, and eliminate one of the variables y or x, as would be done if $\frac{d}{dt}$ were an ordinary constant. This will be done if we multiply the first equation by $\frac{d}{dt}$ and the second by a, and subtract, when we obtain

$$\left(\frac{d^2}{dt^2} - ab\right)x = 0.$$

(It is to be observed, that the word "multiply" is used, not because the operation is really multiplication, since that is a numerical operation, but because it bears a close analogy to multiplication, and is represented symbolically in the same manner.)

Having now eliminated y, we may integrate the equation in x at once. The result is

$$x = c_1\varepsilon^{(ab)^{\frac{1}{2}}t} + c_2\varepsilon^{-(ab)^{\frac{1}{2}}t}.$$

And from the first equation we get

$$y = c_2\left(\frac{b}{a}\right)^{\frac{1}{2}}\varepsilon^{-(ab)^{\frac{1}{2}}t} - c_1\left(\frac{b}{a}\right)^{\frac{1}{2}}\varepsilon^{(ab)^{\frac{1}{2}}t}.$$

As another example, take the equations

$$\frac{dx}{dt} + ax + by = 0,$$

$$\frac{dy}{dt} + a_1x + b_1y = 0;$$

which may be put under the form

$$a\left(\frac{d}{dt} + a\right)x + by = 0,$$

$$a_1x + \left(\frac{d}{dt} + b_1\right)y = 0.$$

Multiply the first by $\frac{d}{dt} + b_1$, and the second by b, and subtract;

$$\text{then } \left\{\left(\frac{d}{dt} + a\right)\left(\frac{d}{dt} + b_1\right) - a_1b\right\}x = 0.$$

The coefficient of x is obviously of the second order. Let it be made up of two factors, so that

$$\left(\frac{d}{dt} + h\right)\left(\frac{d}{dt} + k\right)x = 0,$$

$$\text{where } h + k = a + b_1, \quad hk = ab_1 - a_1b;$$

so that they are the roots of the equation

$$z^2 - (a + b_1)z + ab_1 - a_1b = 0.$$

Integrating in the usual manner,

$$x = c_1\varepsilon^{-ht} + c_2\varepsilon^{-kt};$$

and from the first equation we deduce

$$y = \frac{h - a}{b}c\varepsilon^{-ht} + \frac{k - a}{b}c_1\varepsilon^{-kt}.$$

It will make no difference in the method of the solution, if there be a function of t on the other side of the equation. As, for instance, we have

$$\frac{dx}{dt} + 4x + 3y = t,$$

$$\frac{dy}{dt} + 2x + 5y = \varepsilon^t,$$

$$\text{or } \left(\frac{d}{dt} + 4\right)x + 3y = t,$$

$$\left(\frac{d}{dt} + 5\right)y + 2x = \varepsilon^t.$$

Eliminating y,

$$\left\{\left(\frac{d}{dt} + 4\right)\left(\frac{d}{dt} + 5\right) - 6\right\}x = 1 + 5t - 3\varepsilon^t,$$

$$\text{or } \left(\frac{d}{dt} + 2\right)\left(\frac{d}{dt} + 7\right) x = 1 + 5t - 3\varepsilon^t.$$

Integrating with regard to $\frac{d}{dt} + 7$,

$$\left(\frac{d}{dt} + 2\right) x = \frac{2}{49} + \frac{5}{7} t - \frac{3}{8} \varepsilon^t + c_1 \varepsilon^{-7t},$$

$$\text{and } x = -\frac{31}{196} + \frac{5}{14} t - \frac{1}{8} \varepsilon^t + c_1 \varepsilon^{-7t} + c_2 \varepsilon^{-2t},$$

the value of y will be found to be

$$y = \frac{9}{98} - \frac{1}{7} t + \frac{5}{24} \varepsilon^t - \frac{8}{3} \varepsilon^{-7t} - c_2 \varepsilon^{-2t}.$$

Take also the equations

$$\frac{dx}{dt} + 5x + y = \varepsilon^t,$$

$$\frac{dy}{dt} + 3y - x = \varepsilon^{2t}.$$

Eliminating y, we obtain

$$\left(\frac{d}{dt} + 4\right)^2 x = 4\varepsilon^t - \varepsilon^{2t};$$

$$\text{whence } x = \left(\frac{d}{dt} + 4\right)^{-2} (4\varepsilon^t - \varepsilon^{2t}) + (c_1 t + c_2)\, \varepsilon^{-4t},$$

$$\text{or } x = \frac{4}{25} \varepsilon^t - \frac{1}{36} \varepsilon^{2t} + (c_1 t + c_2)\, \varepsilon^{-4t},$$

$$\text{and } y = \frac{1}{25} \varepsilon^t + \frac{7}{36} \varepsilon^{2t} + (c_1 t + c_2)\, \varepsilon^{-4t}.$$

If there be three simultaneous equations, the same method is to be pursued, but the calculations must be necessarily long, as in ordinary elimination. We may, however, avail ourselves of the method of cross multiplication. If, for instance, we have

$$\frac{dx}{dt} + by + cz = 0,$$

$$\frac{dy}{dt} + a'x + c'z = 0,$$

$$\frac{dz}{dt} + a'x + b'y = 0,$$

$$\text{or } \frac{d}{dt} x + by + cz = 0 \text{ ..... (1)},$$

$$ax + \frac{d}{dt} y + c'z = 0 \text{ ..... (2)},$$

$$a'x + b'y + \frac{d}{dt} z = 0 \text{ .... (3)},$$

we may eliminate y and z by multiplying

(1) by $\frac{d^2}{dt^2} - b'c'$, (2) by $b'c - b\frac{d}{dt}$, (3) by $bc' - c\frac{d}{dt}$,

and the result will be

$$\left\{\frac{d}{dt}\left(\frac{d^2}{dt^2} - b'c'\right) + a\left(b'c - b\frac{d}{dt}\right) + a'\left(bc' - c\frac{d}{dt}\right)\right\} x = 0,$$

$$\text{or } \left\{\frac{d^3}{dt^3} - (ab + a'c + b'c')\frac{d}{dt} + ab'c + a'bc'\right\} x = 0,$$

which can be integrated by the usual method. The equations for determining y and z will, of course, be symmetrical with that for x.

There will be three arbitrary constants arising from the integration of this equation; and it would appear, that as there are other two similar equations, from which six more arbitrary constants would arise, there would be nine on the whole. But there are really only three independent arbitrary constants, as we are able to deduce the other two variables y and z from the value of the first, without integration, and consequently the arbitrary constants in their expressions must be derivable from those in the first integral.

The same method of integration may of course be applied to any number of simultaneous differential equations; but the difficulty of elimination rises rapidly with the number of variables, as in the case of ordinary numerical equations. We shall take as an example of four simultaneous equations, those given by Mr. Airy for determining the secular variations of the eccentricity and longitude of the perihelion, *Plan. Theor.*, p. 123.

They are of the form

$$0 = \frac{d}{dt}u + a_1 v - a_2 v' \ldots\ldots\ldots\ldots (1),$$

$$0 = a_1 u - \frac{d}{dt}v - a_2 u' \ldots\ldots\ldots\ldots (2),$$

$$0 = \frac{d}{dt}u' + b_1 v' - b_2 v \ldots\ldots\ldots\ldots (3),$$

$$0 = b_1 u' - \frac{d}{dt}v' - b_2 u \ldots\ldots\ldots\ldots (4).$$

To reduce them to three simultaneous equations, eliminate v by multiplying (1) by $\frac{d}{dt}$ and (2) by a_1, and adding. Then

$$\left(\frac{d^2}{dt^2} + a_1^2\right) u - a_1 a_2 u' - a_2 \frac{d}{dt} v' = 0 \ldots\ldots\ldots\ldots (5).$$

Again, eliminate the same variable between (1) and (3), by multiplying (1) by b_2, and (3) by a_1, and adding, so that

$$b_2 \frac{d}{dt}u + a_1 \frac{d}{dt}u' + (a_1 b_1 - a_2 b_2) v' = 0 \ldots\ldots\ldots\ldots (6),$$

and the equation (4) is

$$b_2 u - b_1 u' + \frac{d}{dt} v' = 0 \quad \ldots\ldots\ldots\ldots \quad (7).$$

Eliminating u' and v' by cross multiplication between (5), (6), and (7), we obtain an equation which reduces itself to

$$\left\{\frac{d^4}{dt^4} + (a_1^2 + b_1^2 + 2a_2 b_2)\frac{d^2}{dt^2} + a_1 b_1 - a_2 b_2\right\} u = 0 \quad \ldots\ldots \quad (8),$$

which we shall have no difficulty in integrating.

From its form we see that the operating factor is decomposable into two binomial quadratic factors, so that we may put it under the form

$$\left(\frac{d^2}{dt^2} + k_1^2\right)\left(\frac{d^2}{dt^2} + k_2^2\right) u = 0,$$

$-k_1^2$, $-k_2^2$ being the roots of the equation

$$z^2 + (a_1^2 + b_1^2 + 2a_2 b_2)\, z + a_1 b_1 - a_2 b_2 = 0, \quad \ldots\ldots\ldots \quad (9).$$

Integrating with respect to the first factor, as in page 28 of our first Number,

$$\left(\frac{d^2}{dt^2} + k_2^2\right) u = c_1 \cos k_1 x + c_2 \sin k_1 x,$$
$$= c_1 \cos (k_1 x + a),$$

by changing the form of the constants; whence

$$u = \left(\frac{d^2}{dt^2} + k_2^2\right)^{-1} \{c_1 \cos (k_1 x + a_1)\} + c_2 \cos (k_2 x + a_2).$$

The operating factor will only affect the constant of its subject, and as that is arbitrary, we may write

$$u = c_1 \cos (k_1 x + a_1) + c_2 \cos (k_2 x + a_2).$$

To find the value of v, we may observe, that if we had eliminated v' instead of v at first, we should have, instead of (6) and (7), the similar equations

$$a_2 \frac{d}{dt} u' + b_1 \frac{d}{dt} u + (a_1 b_1 - a_2 b_2)\, v = 0,$$

$$a_2 u' - a_1 u + \frac{d}{dt} v = 0.$$

Eliminating u' between these by multiplying the second by $\frac{d}{dt}$, and subtracting,

$$\left\{\frac{d^2}{dt^2} - (a_1 b_1 - a_2 b_2)\right\} v = (a_1 + b_1)\frac{du}{dt};$$

or putting for u its value, and effecting the differentiation indicated,

$$\left\{\frac{d^2}{dt^2} - (a_1 b_1 - a_2 b_2)\right\} v$$
$$= -(a_1 + b_1)\{c_1 k_1 \sin (k_1 x + a_1) + c_2 k_2 \sin (k_2 x + a_2)\};$$

whence, putting $a_1b_1 - a_2b_2 = A$,

$$v=(a_1+b_1)\left\{\frac{c_1k_1}{k_1^2+A}\sin(k_1x+a_1)+\frac{c_2k_2}{k_2^2+A}\sin(k_2x+a_2)\right\}.$$

By substituting the actual expressions for k_1 and k_2, this may be simplified. For solving equation (9), we get

$$k_1^2=\tfrac{1}{2}\{a_1^2+b_1^2+2a_2b_2-(a_1+b_1)[(a_1-b_1)^2+4a_2b_2]^{\frac{1}{2}}\};$$

then $k_1^2+A=\tfrac{1}{2}(a_1+b_1)\{a_1+b_1-[(a_1-b_1)^2+4a_2b_2]^{\frac{1}{2}}\}$.

But we have also

$$k_1=\tfrac{1}{2}\{a_1+b_1-[(a_1-b_1)^2+4a_2b_2]^{\frac{1}{2}}\},$$

and similarly for k_2; so that

$$v=c_1\sin(k_1x+a_1)+c_2\sin(k_2x+a_2).$$

Knowing u and v, we easily determine u' and v'. For, by equation (2),

$$a_2u'=a_1u-\frac{d}{dt}v$$

$$=(a_1-k_1)c_1\cos(k_1x+a_1)+(a_1-k_2)c_2\cos(k_2x+a_2);$$

similarly v' may be found.

In this case we have had an example of the integration of simultaneous differential equations, of an order higher than the first. We shall take another in

$$\frac{d^2y}{dt^2}-az-by=c,$$

$$\frac{d^2z}{dt^2}-a'z-b'y=c',$$

being two of the equations for determining the circumstances of the movement of a floating body in a position nearly of equilibrium, the other two being

$$\frac{d^2\phi}{dt^2}+n\phi-m=0,$$

$$\frac{d^2s}{dt^2}-g\frac{d^2\phi}{dt^2}=0.$$

Putting the equations under the form

$$\left(\frac{d^2}{dt^2}-b\right)y-az=c,$$

$$\left(\frac{d^2}{dt^2}-a'\right)z-b'y=c',$$

we can eliminate z by multiplying the first by $\frac{d^2}{dt^2}-a'$, and the second by a, and adding, when we get

$$\left(\frac{d^2}{dt^2}-a'\right)\left(\frac{d^2}{dt^2}-b\right)y-ab'y=ac'-ca'.$$

If k_1^2, k_2^2 be the roots of

$$z^2 - (a' + b)\, z + a'b - ab' = 0,$$

the equation becomes

$$\left(\frac{d^2}{dt^2} - k_1^2\right)\left(\frac{d^2}{dt^2} - k_2^2\right) y = ac' - ca';$$

the integral of which is

$$y = \frac{ac' - ca'}{k_1^2\, k_2^2} + c_1\, \varepsilon^{k_1 t} + c_2\, \varepsilon^{-k_1 t} + c_3\, \varepsilon^{k_2 t} + c_4\, \varepsilon^{-k_2 t},$$

$$y = \frac{ac' - ca'}{a'b - ab'} + c_1\, \varepsilon^{k_1 t} + c_2\, \varepsilon^{-k_1 t} + c_3\, \varepsilon^{k_2 t} + c_4\, \varepsilon^{-k_2 t}.$$

The value of z may be found from that of y; and if we know the initial circumstances, we can determine the arbitrary constants.

The same method may be extended to simultaneous partial differential equations, according to the principles developed in Article IX. of our third Number. Take, for instance, the equations

$$\frac{dz}{dx} + c\,\frac{du}{dx} + a\,\frac{dz}{dy} + bz = 0,$$

$$c'\,\frac{dz}{dx} + \frac{du}{dx} + a\,\frac{du}{dy} + bu = 0,$$

which may be written

$$\left(\frac{d}{dx} + a\,\frac{d}{dy} + b\right) z + c\,\frac{d}{dx}\,u = 0,$$

$$c'\,\frac{d}{dx}\,z + \left(\frac{d}{dx} + a\,\frac{d}{dy} + b\right) u = 0.$$

To eliminate u, multiply the first by $\left(\frac{d}{dx} + a\,\frac{d}{dy} + b\right)$, and the second by $c'\,\frac{d}{dx}$, and subtract. Then

$$\left\{\frac{d^2}{dx^2} + \frac{2\left(a\,\frac{d}{dy} + b\right)}{1 - cc'}\,\frac{d}{dx} + \frac{\left(a\,\frac{d}{dy} + b\right)^2}{1 - cc'}\right\} z = 0.$$

If we put $1 - \sqrt{cc'} = m_1$, $1 + \sqrt{cc'} = m_2$, this may be resolved into factors,

$$\left\{\frac{d}{dx} + m_1\left(a\,\frac{d}{dy} + b\right)\right\}\left\{\frac{d}{dx} + m_2\left(a\,\frac{d}{dy} + b\right)\right\} z = 0;$$

the integral of which is

$$z = \varepsilon^{-m_1 x\left(a\,\frac{d}{dy} + b\right)}\,\phi(y) + \varepsilon^{-m_2 x\left(a\,\frac{d}{dy} + b\right)}\,\psi(y),$$

or $z = \varepsilon^{-m_1 bx}\,\phi(y - am_1 x) + \varepsilon^{-m_2 bx}\,\psi(y - am_2 x)$.

Mr. Airy, at page 279 note of the Undulatory Theory, gives as

the equations for determining the small disturbances of an elastic medium in three dimensions,

$$\frac{d^2u}{dt^2} = a^2 \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$
$$\frac{d^2v}{dt^2} = a^2 \frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$
$$\frac{d^2w}{dt^2} = a^2 \frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

We might eliminate v and w, by cross multiplication, between these equations, and so obtain an equation in u which might be integrated, but it will be more convenient to proceed as follows. Let

$$r = a^2\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right);$$

$$\text{then } \frac{d^{-1}}{dx^{-1}}\frac{d^2u}{dt^2} = \frac{d^{-1}}{dy^{-1}}\frac{d^2v}{dt^2} = \frac{d^{-1}}{dz^{-1}}\frac{d^2w}{dt^2} = r.$$

Multiply by $\frac{d^2}{dx^2}$, $\frac{d^2}{dy^2}$, $\frac{d^2}{dz^2}$, and add; then

$$\frac{d^2}{dt^2}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) r,$$

$$\text{or } \left\{\frac{d^2}{dt^2} - a^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\right\} r = 0;$$

the integral of which is

$$r = \varepsilon^{at\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^{\frac{1}{2}}}\phi(x, y, z) + \varepsilon^{-at\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^{\frac{1}{2}}}\psi(x, y, z);$$

knowing r, we can determine u, v, and w, since

$$u = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dx}, \quad v = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dy}, \quad w = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dz}.$$

This solution is due to Mr. Greatheed.

We have now given a sufficient number of examples to enable the student to understand thoroughly the method, and we think that they show clearly the advantages of a process, which, to some persons, might appear to carry out to a startling extent the principles on which it is founded.

D. F. G.

IX.—ANALYTICAL SOLUTIONS OF SOME PROBLEMS IN PLANE ASTRONOMY.

When it has been required to determine the ratio of the small variations of arcs or angles on the surface of a sphere, mathematicians have generally employed a geometrical method, probably because it appeared to them that the application of the Differential Calculus would lead to laborious processes. But it will be shewn in this Article, that Analytical solutions of such problems are not necessarily tedious, and the method by which they are abridged will be seen to be very similar in all.

1. When the latitude and hour-angle are determined from two altitudes of the Sun, and the time between, to find the errors caused by given small errors in the observed altitudes.

Let l be the latitude, z one of the observed zenith distances, h the hour-angle from noon, δ the Sun's declination, a his south azimuth. Then

$$\cos z = \sin l \sin \delta + \cos l \cos \delta \cos h.$$

Differentiate, using Δ to denote small finite variations, which will be nearly in the ratio of the differentials; and considering δ constant, as the Sun's declination is supposed to be known accurately: therefore

$$-\sin z \Delta z = (\cos l \sin \delta - \sin l \cos \delta \cos h)\, \Delta l \quad \cos l \cos \delta \sin h\, \Delta h.$$

$$\text{Now} \quad \cos l \sin \delta - \sin l \cos \delta \cos h = -\cos a \sin z,$$

$$\text{and} \quad \cos \delta \sin h = \sin z \sin a;$$

substituting and dividing by $\sin z$,

$$\Delta z = \cos a \Delta l + \cos l \sin a \Delta h.$$

If z', a' be the zenith distance and azimuth at the second observation,

$$\Delta z' = \cos a' \Delta l + \cos l \sin a' \Delta h;$$

$$\text{therefore} \quad \Delta l = \frac{\Delta z \sin a' - \Delta z' \sin a}{\sin (a' - a)},$$

$$\text{and} \quad \Delta h = \frac{\Delta z' \cos a - \Delta z \cos a'}{\cos l \sin (a' - a)}.$$

2. To find the errors caused by parallax in the hour-angle and declination of a heavenly body.

The notation remaining the same, we have the equation

$$\cot h \sin a = \cos l \cot z + \sin l \cos a,$$

in which h and z are the only variables. Differentiating,

$$\frac{\sin a}{(\sin h)^2} \Delta h = \frac{\cos l}{(\sin z)^2} \Delta z.$$

But if P be the horizontal parallax,

$$\Delta z = \mathrm{P} \sin z,$$

$$\text{and } \sin a = \frac{\cos \delta \sin h}{\sin z};$$

$$\text{therefore } \Delta h = \mathrm{P} \frac{\cos l \sin h}{\cos \delta}.$$

$$\text{Again, } -\cos a = \frac{\sin \delta - \sin l \cos z}{\cos l \sin z},$$

$$\text{or } \cos a \cos l = \sin l \cot z - \sin \delta \operatorname{cosec} z.$$

Differentiating,

$$0 = \{\sin \delta \operatorname{cosec} z \cot z - \sin l (\operatorname{cosec} z)^2\} \Delta z - \cos \delta \operatorname{cosec} z \, \Delta \delta,$$

$$0 = \mathrm{P} (\sin \delta \cos z - \sin l) - \cos \delta \, \Delta \delta,$$

$$\text{but } \cos z = \sin l \sin \delta + \cos l \cos \delta \cos h;$$

$$\text{therefore } \cos \delta \, \Delta \delta = \mathrm{P} \{\sin l (\sin \delta)^2 + \cos l \sin \delta \cos \delta \cos h - \sin l\},$$

$$\text{and } \Delta \delta = \mathrm{P} (\cos l \sin \delta \cos h) - \sin l \cos \delta).$$

3. To find the precession of a star in right ascension and declination.

Let a, δ, l, λ, be the right ascension, declination, longitude, and latitude of the star, and ω the obliquity of the ecliptic.

$$\text{Then } \sin \delta = \cos \omega \sin \lambda + \sin \omega \cos \lambda \sin l;$$

$$\text{therefore } \cos \delta \, \Delta \delta = \sin \omega \cos \lambda \cos l \, \Delta l.$$

$$\text{But } \cos \lambda \cos l = \cos \delta \cos a;$$

$$\text{therefore } \Delta \delta = \sin \omega \cos a \, \Delta l,$$

$$= 50'',18.t. \sin \omega \cos a,$$

t being the interval expressed in years.

$$\text{Again, } \sin \lambda = \cos \omega \sin \delta - \sin \omega \cos \delta \sin a;$$

therefore

$$0 = (\cos \omega \cos \delta + \sin \omega \sin \delta \sin a) \Delta \delta - \sin \omega \cos \delta \cos a \, \Delta a.$$

Substituting the value of $\Delta \delta$,

$$\Delta a = 50'',18.t. (\cos \omega + \sin \omega \tan \delta \sin a).$$

4. To find the aberration of a star in longitude and latitude.

Let S (fig. 4.) be a star, O a point in the ecliptic 90° behind the Earth's plane, SL perpendicular to the ecliptic.

Let l, λ be the latitude and longitude of the star. Then

$$\Upsilon \mathrm{O} = 180^0 + \odot - 90^0 = 90^0 + \odot,$$

$$\mathrm{LO} = 90^0 + \odot - l.$$

By Napier's rules,

$$\cos \mathrm{O} = \cot \mathrm{OS} \tan \mathrm{LO},$$

$$= -\cot \mathrm{OS} \cot (\odot - l).$$

Taking the logarithmic differentials,

$$0 = \frac{\Delta.\mathrm{OS}}{\sin \mathrm{OS} \cos \mathrm{OS}} - \frac{\Delta l}{\sin (\odot - l) \cos (\odot - l)}.$$

But $\Delta.\mathrm{OS} = n \sin \mathrm{OS}$; n denoting the ratio of the Earth's velocity to that of light, and

$$\cos \mathrm{OS} = \cos \mathrm{SL}.\cos \mathrm{LO} = -\cos \lambda \sin (\odot - l).$$

$$\text{Hence } \Delta l = -n \frac{\cos (\odot - l)}{\cos \lambda}.$$

$$\text{Again, } \sin \lambda = \sin \mathrm{O}.\sin \mathrm{OS};$$

$$\text{therefore } \cot \lambda \Delta\lambda = \cot \mathrm{OS}.\Delta.\mathrm{OS},$$

$$= n \cos \mathrm{OS},$$

$$= -n \cos \lambda \sin (\odot - l);$$

$$\text{therefore } \Delta\lambda = -n \sin \lambda \sin (\odot - l).$$

5. The aberration in right ascension and declination may be found in the following manner.

$$\text{Since } \tan a = \cos \omega \tan l - \sin \omega \sec l \tan \lambda,$$

$$(\sec a)^2 \Delta a = \cos \omega (\sec l)^2 \Delta l - \sin \omega \{\sec l \tan l \tan \lambda \Delta l + \sec l (\sec \lambda)^2 \Delta\lambda\}$$

$$= -n \cos \omega (\sec l)^2 \sec \lambda \cos (\odot - l) + n \sin \omega \sec l \sec \lambda \tan \lambda \{\tan l \cos (\odot - l) + \sin (\odot - l)\},$$

$$= -n \cos \omega (\sec l)^2 \sec \lambda \cos (\odot - l) + n \sin \omega (\sec l)^2 \sec \lambda \tan \lambda \sin \odot,$$

$$= -n \frac{\cos \omega \cos \odot + (\cos \omega \tan l - \sin \omega \sec l \tan \lambda) \sin \odot}{\cos l \cos \lambda},$$

$$= -n \frac{\cos \omega \cos \odot + \tan a \sin \odot}{\cos a \cos \delta};$$

$$\text{therefore } \Delta a = -n \frac{\cos \omega \cos a \cos \odot + \sin a \sin \odot}{\cos \delta}.$$

Also, since $\sin \delta = \sin \omega \sin l \cos \lambda + \cos \omega \sin \lambda$,

$$\cos \delta \Delta\delta = \sin \omega (\cos l \cos \lambda \Delta l - \sin l \sin \lambda \Delta\lambda) + \cos \omega \cos \lambda \Delta\lambda,$$

$$= -n \sin \omega \{\cos l \cos (\odot - l) - \sin l (\sin \lambda)^2 \sin (\odot - l)\} - n \cos \omega \sin \lambda \cos \lambda \sin (\odot - l)\},$$

$$= -n \sin \omega \cos \odot - n (\sin \omega \sin l \cos \lambda + \cos \omega \sin \lambda) \cos \lambda \sin (\odot - l),$$

$$= -n \sin \omega \cos \odot - n \sin \delta \cos \lambda \sin (\odot - l),$$

$$= -n \sin \omega \cos \odot - n \sin \delta \cos l \cos \lambda (\sin \odot - \cos \odot \tan l),$$

$$= -n \sin \omega \cos \odot - n \sin \delta \cos a \cos \delta \{\sin \odot - \cos \odot (\cos \omega \tan a + \sin \omega \tan \delta \sec a)\},$$

$$= -n \sin \omega (\cos \delta)^2 \cos \odot - n \sin \delta \cos \delta (\cos a \sin \odot - \cos \omega \sin a \cos \odot);$$

therefore $\Delta a = -n\{\sin\omega\cos\delta\cos\odot + \sin\delta(\cos a\sin\odot - \cos\omega\sin a\cos\odot)\}$.

6. To find the time of year at which the Sun passes from one given zenith distance to another in the shortest or longest time.

Let Z (fig 5.) be the zenith, P the pole of the equator, S, S′ the two positions of the Sun. Let z, z' be the given zenith distances, h, h' the corresponding hour angles, δ the Sun's declination, and l the latitude of the place. Then

$$\cos z = \sin l\sin\delta + \cos l\cos\delta\cos h \ldots\ldots\ldots (1),$$
$$\cos z' = \sin l\sin\delta + \cos l\cos\delta\cos h' \ldots\ldots\ldots (2),$$

and $h' - h$ is a maximum or minimum; therefore

$$0 = (\sin l\cos\delta - \cos l\sin\delta\cos h)\,d\delta - \cos l\cos\delta\sin h\,dh,$$
$$0 = (\sin l\cos\delta - \cos l\sin\delta\cos h')\,d\delta - \cos l\cos\delta\sin h'\,dh',$$
$$dh' - dh = 0;$$

hence $$\left(\frac{\sin l\cos\delta - \cos l\sin\delta\cos h'}{\sin h'} - \frac{\sin l\cos\delta - \cos l\sin\delta\cos h}{\sin h}\right)d\delta = 0 \ldots\ldots (3);$$

therefore, either $d\delta = 0$, which gives the times of the solstices, or the other factor is equal to zero. Now

$$\frac{\sin l\cos\delta - \cos l\sin\delta\cos h}{\cos l\sin h} = \cot \mathrm{PSZ};$$

therefore $\cot \mathrm{PSZ} = \cot \mathrm{PS'Z}$,

therefore $\cos \mathrm{PSZ} = \pm \cos \mathrm{PS'Z}$,

or $$\frac{\sin l - \cos z\sin\delta}{\sin z\cos\delta} = \pm\frac{\sin l - \cos z'\sin\delta}{\sin z'\cos\delta},$$

$$\sin l(\sin z' \mp \sin z) = \sin\delta\sin(z' \mp z) \ldots\ldots\ldots (4).$$

Taking the upper sign, we have

$$2\sin l\cos\frac{z'+z}{2}\sin\frac{z'-z}{2} = 2\sin\delta\sin\frac{z'-z}{2}\cos\frac{z'-z}{2};$$

$$\text{therefore}\quad \sin\delta = \sin l\frac{\cos\frac{z'+z}{2}}{\cos\frac{z'-z}{2}} \ldots\ldots\ldots (5);$$

whence the time of year is known.

If it be required to find the time between the zenith distances, we have from (3)

$$\sin l\cos\delta(\sin h' - \sin h) = \cos l\sin\delta\sin(h' - h);$$

$$\text{therefore}\quad \sin l\cos\delta\cos\frac{h'+h}{2} = \cos l\sin\delta\cos\frac{h'-h}{2}.$$

By adding (1) and (2),

$$\cos\frac{z'+z}{2}\cos\frac{z'-z}{2} = \sin l \sin\delta + \cos l \cos\delta \cos\frac{h'+h}{2}\cos\frac{h'-h}{2}.$$

Eliminating $\cos\dfrac{h'+h}{2}$,

$$\sin l \cos\frac{z'+z}{2}\cos\frac{z'-z}{2} = \sin\delta\left\{(\sin l)^2 + (\cos l)^2 \cos\left(\frac{h'-h}{2}\right)^2\right\}.$$

Substituting the value of $\sin\delta$ from (5),

$$\left(\cos\frac{z'-z}{2}\right)^2 = (\sin l)^2 + (\cos l)^2 \cos\left(\frac{h'-h}{2}\right)^2;$$

therefore $\left(\sin\dfrac{z'-z}{2}\right)^2 = (\cos l)^2\left(\sin\dfrac{h'-h}{2}\right)^2$,

$$\sin\frac{h'-h}{2} = \pm \sec l \sin\frac{z'-z}{2} \quad \ldots\ldots\ldots\ldots (6).$$

To find the results of taking the lower sign in (4), we have only to change the sign of z, which gives

$$\sin\delta = \sin l \frac{\cos\dfrac{z'-z}{2}}{\cos\dfrac{z'+z}{2}} \quad \ldots\ldots\ldots\ldots (7),$$

$$\sin\frac{h'-h}{2} = \pm \sec l \sin\frac{z'+z}{2} \quad \ldots\ldots\ldots\ldots (8).$$

Since there are two periods in the year at which the Sun has the same declination, within the same half year, and at equal distances from the equinoxes, equations (5) and (7) give each two periods of maximum or minimum duration, and the solstices are two others, so that there are six in all. But one or both of the equations (5) and (7) may give a value of δ greater than the obliquity of the ecliptic, which, therefore, the Sun never attains, so that there may be only four or two maxima and minima. In the problem of finding the time of longest or shortest twilight, the values of z and z' are 90^0 and 108^0; so that equations (5) and (7) become respectively

$$\sin\delta = -\sin l \tan 9^0 \quad \ldots\ldots\ldots\ldots\ldots (9),$$

$$\sin\delta = -\sin l \cot 9^0 \quad \ldots\ldots\ldots\ldots\ldots (10).$$

Of these, the former gives a value of δ less than the obliquity, whatever be the latitude. The value for north latitude, 52^0, is $-7^0.10'.10''$, which occurs on the 2nd or 3rd of March, and the 11th or 12th of October. But (10) will give a value of δ which is never attained, unless l be not greater than $3^0.36'.55''$. At the

equator the values of δ given by (9) and (10) coincide, being both zero.

After some considerations, which enable us to distinguish between maximum and minimum values, we obtain the following results.

At the equator, the duration of twilight is a minimum at the equinoxes, and a maximum at the solstices.

Between the equator and latitude $3^{0}.36'.55''$, the duration is a minimum at two periods near the equinoxes, in the winter half of the year, a maximum at two others nearer the winter solstice, a minimum at the winter, and a maximum at the summer solstice.

At and above latitude $3^{0}.36'.55''$, the duration is a minimum at two periods near the equinoxes, in the winter half of the year, and a maximum at the solstices.

The seasons are here spoken of, taking into account their being opposite on different sides of the equator.

C. G.

X.—ON CERTAIN CASES OF CONSECUTIVE SURFACES.*

THERE is a very simple theorem, which does not seem to have been much noticed by English writers, but which is of frequent use in Analytical Geometry, particularly when all the equations are put in a symmetrical form. It is, that the formula

$$\frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''} = \ldots$$

always implies the truth of the following, viz.

$$\frac{a}{b} = \frac{a'}{b'} = \ldots = \frac{a + a' + a'' + \ldots}{b + b' + b'' + \ldots} = \frac{ma + m'a' + m''a'' + \ldots}{mb + m'b' + m''b'' + \ldots}$$

$$= \frac{\sqrt{a^2 + a'^2 + a''^2 + \ldots}}{\sqrt{b^2 + b'^2 + b''^2 + \ldots}},$$

m, m', m'', ... being any quantities whatever. The proof is so obvious, that it is unnecessary to put it down here. This being premised, suppose the equation to a surface is

$$u = h \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where h is an absolute constant, and $u = \mathrm{F}(x, y, z, a, b, c)$, a, b, c being parameters, by the variation of which the surface

---

* From a Correspondent.

takes its different positions. And first, suppose a, b, and c to vary, subject to the single condition

$$f(a, b, c) = k \ldots\ldots\ldots\ldots (2),$$

which is equivalent to supposing the equation (1) to contain two arbitrary parameters. Now, suppose the partial differential coefficients of u and $f(a, b, c)$, taken with regard to a, b, c, are P, Q, R, A, B, C, respectively. Then, in order to find the intersection of the surface with its consecutives, we shall obviously have to combine the equation (1) with the two following, viz.

$$\mathrm{P}da + \mathrm{Q}db + \mathrm{R}dc = 0,$$
$$\mathrm{A}da + \mathrm{B}db + \mathrm{C}dc = 0.$$

If between these two equations we eliminate da, we obtain

$$(\mathrm{AQ} - \mathrm{BP})\, db + (\mathrm{AR} - \mathrm{PC})\, dc = 0;$$

and since there is no relation between db and da, their coefficients in this equation must separately be equal to nothing; that is,

$$\mathrm{AQ} - \mathrm{BP} = 0, \quad \mathrm{AR} - \mathrm{PC} = 0,$$

whence we immediately derive

$$\frac{\mathrm{P}}{\mathrm{A}} = \frac{\mathrm{Q}}{\mathrm{B}} = \frac{\mathrm{R}}{\mathrm{C}} \ldots\ldots\ldots\ldots (3).$$

This formula, combined with (1), determines the coordinates of the point in which the surface is intersected by all its consecutives; and if we also combine it with (2), and eliminate a, b, and c, we shall obtain the equation to the envelope. This elimination will often be easy, if the functions u and $f(a, b, c)$ are homogeneous with respect to a, b, and c. For suppose m and n are their respective degrees of homogeneity, and we shall have

$$a\mathrm{P} + b\mathrm{Q} + c\mathrm{R} = mh,$$
$$a\mathrm{A} + b\mathrm{B} + c\mathrm{C} = nk.$$

Hence, applying the theorem above laid down to the formula (3), we get

$$\frac{\mathrm{P}}{\mathrm{A}} = \frac{\mathrm{Q}}{\mathrm{B}} = \frac{\mathrm{R}}{\mathrm{C}} \left(= \frac{a\mathrm{P} + b\mathrm{Q} + c\mathrm{R}}{a\mathrm{A} + b\mathrm{B} + c\mathrm{C}}\right) = \frac{mh}{nk} \ldots\ldots\ldots (4),$$

by the help of which a, b, and c may be eliminated from either of the equations (1), (2).

If the generating surface was a plane, suppose its equation was

$$ax + by + cz = h^2 \ldots\ldots\ldots\ldots (5),$$

and we should have $\mathrm{P} = x$, $\mathrm{Q} = y$, $\mathrm{R} = z$; and hence, substituting in the formula (3) and applying the same theorem, we should get

$$\frac{x}{\mathrm{A}} = \frac{y}{\mathrm{B}} = \frac{z}{\mathrm{C}} = \frac{h^2}{a\mathrm{A} + b\mathrm{B} + c\mathrm{C}} = \frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2}} \ldots\ldots\ldots (6).$$

These equations determine immediately the coordinates x, y, z of the point in which the plane is intersected by all its consecu-

tives, and the length of the radius vector r from the origin to this point. Also, if λ, μ, ν are the angles which r makes with the three axes, we have evidently

$$\frac{\cos\lambda}{A} = \frac{\cos\mu}{B} = \frac{\cos\nu}{C} = \frac{1}{\sqrt{A^2 + B^2 + C^2}} \quad \ldots\ldots\ldots (7).$$

Hence it is easily seen, that if we described a surface whose equation was

$$f(x, y, z) = k \quad \ldots\ldots\ldots\ldots (8),$$

a, b, and c would always be equal to the coordinates of some point in it; and if a tangent plane was drawn at this point, the perpendicular upon it from the origin would make precisely the same angles with the axes as those indicated by the formula (7), and would therefore coincide with the radius vector r. Moreover, if p was the length of the perpendicular, we should have

$$p = \frac{aA + bB + cC}{\sqrt{A^2 + B^2 + C^2}};$$

and therefore it is evident, from the equations (6), that $rp = h^2$. Hence, if in the perpendicular from the origin upon the tangent plane to the surface (8), we take a point, such that its distance from the origin multiplied by the length of the perpendicular shall $= h^2$, the locus of this point will be the envelope of the system of planes defined by the two equations

$$ax + by + cz = h^2, \quad f(a, b, c) = k.$$

As another example, suppose the equation to the surface is

$$\frac{\phi}{a^m} + \frac{\chi}{b^m} + \frac{\psi}{c^m} = 1 \quad \ldots\ldots\ldots (9),$$

where ϕ, χ, and ψ are any functions of x, y, z, not containing a, b, c; and suppose the relation between a, b, and c is

$$\left(\frac{a}{\alpha}\right)^n + \left(\frac{\beta}{b}\right)^n + \left(\frac{\gamma}{c}\right)^n = 1 \quad \ldots\ldots\ldots (10):$$

here the equations (4) become

$$\frac{\phi}{\alpha^n a^{m-n}} = \frac{\chi}{\beta^n b^{m-n}} = \frac{\psi}{\gamma^n c^{m-n}} = 1 \quad \ldots\ldots\ldots (11);$$

and if we take the values of a, b, c from these, and substitute them in (9) or (10), we get the equation to the envelope, viz.

$$\left(\frac{\phi}{\alpha^m}\right)^{\frac{n}{n-m}} + \left(\frac{\chi}{\beta^m}\right)^{\frac{n}{n-m}} + \left(\frac{\psi}{\gamma^m}\right)^{\frac{n}{n-m}} = 1.$$

In the particular case when $n = m$, the equations (11) are reduced to the three $\phi = \alpha^m$, $\chi = \beta^m$, $\psi = \gamma^m$, and the envelope consists of three distinct surfaces defined by these equations. And if ϕ, χ, ψ each contain only one of the variables x, y, z, the envelope is reduced to a certain definite number of points. For instance, all the ellipsoids defined by the two equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1,$$

will intersect one another in the eight points, whose coordinates are $\pm\alpha$, $\pm\beta$, $\pm\gamma$.

But if a, b, c were subject to two equations of condition, suppose, for instance,

$$f(a, b, c) = k \ldots\ldots\ldots\ldots (12),$$
$$f_1(a, b, c) = k_1 \ldots\ldots\ldots\ldots (13),$$

(which is the same thing as supposing the equation to the surface to contain only one arbitrary parameter,) let A_1, B_1, C_1, be the partial differential coefficients of $f_1(a, b, c)$, the others remaining as before, and we shall have the three equations

$$Pda + Qdb + Rdc = 0,$$
$$Ada + Bdb + Cdc = 0,$$
$$A_1da + B_1db + C_1dc = 0.$$

To eliminate the differentials, apply the method of indeterminate multipliers, and we shall have

$$\left.\begin{aligned} P &= \lambda A + \mu A_1 \\ Q &= \lambda B + \mu B_1 \\ R &= \lambda C + \mu C_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (14).$$

Let $aP+bQ+cR=v$, $aA+bB+cC=S$, $aA_1+bB_1+cC_1=S_1$; then, if we add the equations (14), after having respectively multiplied them by a, b, and c, we get

$$v = \lambda S + \mu S_1;$$

and if we substitute in the equations (14) the value of λ derived from this equation, we obtain

$$SP - Av = \mu(A_1S - AS_1),$$
$$SQ - Bv = \mu(B_1S - BS_1),$$
$$SR - Cv = \mu(C_1S - CS_1),$$

and therefore

$$\frac{SP - Av}{A_1S - AS_1} = \frac{SQ - Bv}{B_1S - BS_1} = \frac{SR - Cv}{C_1S - CS_1} \ldots\ldots\ldots (15).$$

This formula, combined with (1), comprehends the equations to the curve in which the surface is intersected by its consecutive; and if we eliminate a, b, c by means of (12) and (13), we shall obtain the equation to the envelope. If the functions u, $f(a, b, c)$, $f_1(a, b, c)$, are homogeneous, and of the degrees n, m, m_1 respectively, then $v = nh$, $S = mk$, and $S^1 = m_1k_1$.

Suppose, for example, it was required to find the surface generated by the consecutive intersections of tangent planes drawn to the ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (16),$$

at all the points where it is cut by the plane, whose equation is

$$lx + my + nz = k \ldots\ldots\ldots (17).$$

The equation to a tangent plane at the point (x, y, z), is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} = 1 \ldots\ldots\ldots (18),$$

ξ, η, ζ being the coordinates of any point in the tangent plane. Here the variable parameters are x, y, z, subject to the two equations (16) and (17). And we have $P = \frac{\xi}{a^2}$, $A = \frac{2x}{a^2}$, $A_1 = l$, &c., $v = 1$, $S = 2$, $S_1 = k$; hence the formula (15) becomes

$$\frac{\xi - x}{la^2 - kx} = \frac{\eta - y}{mb^2 - ky} = \frac{\zeta - z}{nc^2 - kz} \ldots\ldots\ldots (19),$$

which is equivalent to the equations to the generating line of the envelope. We may put it under another form, by subtracting 1 from each term, after having multiplied by k; we thus obtain

$$\frac{k\xi - la^2}{la^2 - kx} = \frac{k\eta - mb^2}{mb^2 - ky} = \frac{k\zeta - nc^2}{nc^2 - kz}.$$

This shews that the generating line always passes through a fixed point, whose coordinates are $\frac{la^2}{k}, \frac{mb^2}{k}, \frac{nc^2}{k}$.

In order to eliminate x, y, z from the equations (19), if we multiply the numerator and denominator of each term by $\frac{\xi}{a^2}$, $\frac{\eta}{b^2}$, and $\frac{\zeta}{c^2}$ respectively, and then add the numerators and denominators, we obtain an expression which, by the help of equations (17) and (18), immediately reduces to the following, viz:

$$\frac{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1}{l\xi + m\eta + n\zeta - k}.$$

Again, if we multiply by l, m, n, and add, we obtain

$$\frac{l\xi + m\eta + n\zeta - k}{l^2a^2 + m^2b^2 + n^2c^2 - k^2};$$

and these two expressions must be equal, by the principle stated at the beginning. Hence,

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1 + \frac{(l\xi + m\eta + n\zeta - k)^2}{l^2a^2 + m^2b^2 + n^2c^2 - k^2},$$

which is the equation to the conical surface required.

If it was required to find the locus of the vertex of this cone, when the plane which cuts the ellipsoid is moved, but made to pass through a fixed point (α, β, γ), suppose x, y, z are the coordinates of the vertex, and we have seen above that we shall have

$$\frac{x}{la^2} = \frac{y}{mb^2} = \frac{z}{nc^2} = \frac{1}{k} \ldots\ldots\ldots\ldots (20);$$

also, α, β, γ must satisfy the equation to the cutting plane, that is,

$$l\alpha + m\beta + n\gamma = k;$$

multiplying, then, the terms of this last equation by the terms of (20), we have

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

which is the equation to the locus sought, and represents a plane. If the point (α, β, γ) is a point in the surface of the ellipsoid, this equation coincides with the equation to the tangent plane at that point, as it evidently ought.

M. N. N.

XI.—MATHEMATICAL NOTE.

Geometrical Theorem. Let $A_1A_2A_3 \ldots$ be a polygon of n sides inscribed in a circle, a_1, a_2, a_3, &c. the angles which the sides A_1A_2, A_2A_3, &c. subtend at the centre. Then

$$\angle A_1A_2A_3 = \pi - A_1A_5A_3,$$
$$= \pi - \frac{a_1 + a_2}{2},$$
$$\angle A_3A_4A_5 = \pi - \frac{a_3 + a_4}{2},$$
&c. &c.
$$\angle A_{n-1}A_nA_1 = \pi - \frac{a_{n-1} + a_n}{2}.$$

If n be even, adding all these together, we get

$$\angle A_1A_2A_3 + \angle A_3A_4A_5 + \text{&c.} + \angle A_{n-1}A_nA_1$$
$$= \frac{n}{2}\pi - \frac{2\pi}{2} = (n-2)\frac{\pi}{2},$$

or the sum of the alternate angles is equal to $n-2$ right angles, a curious extension of Euclid, III. 22.

γ.

Fig

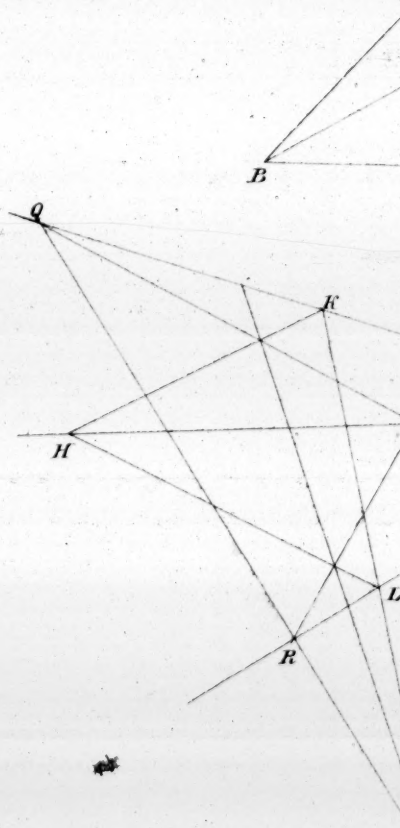


Fig. 4

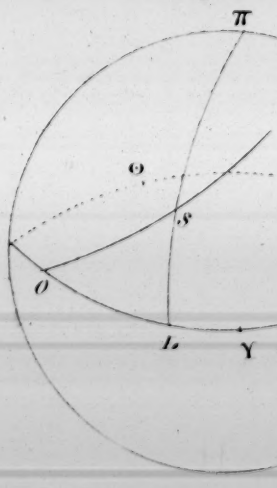


Fig. 1

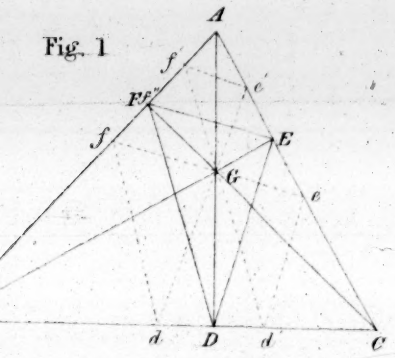
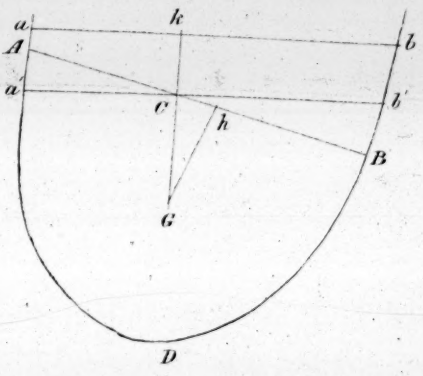


Fig. 2.



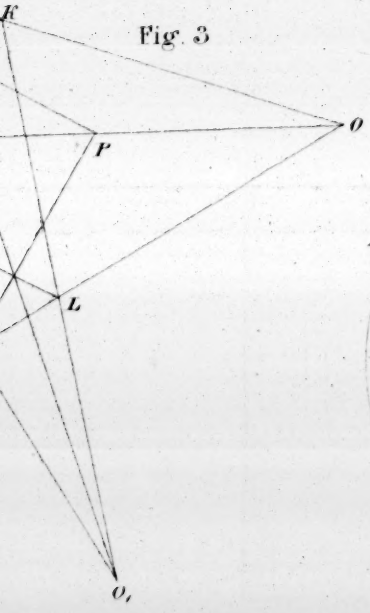
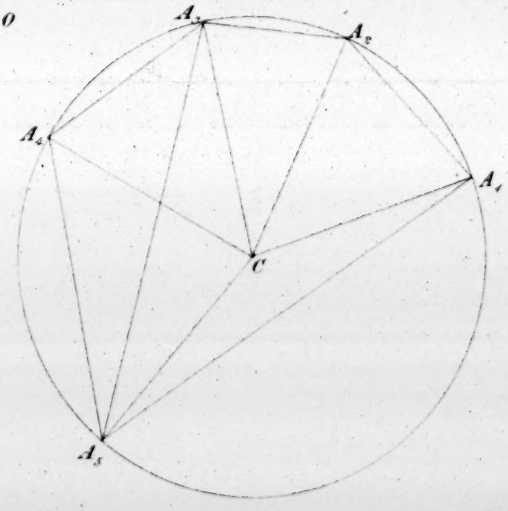


Fig. 3

Fig. 6.



g. 4

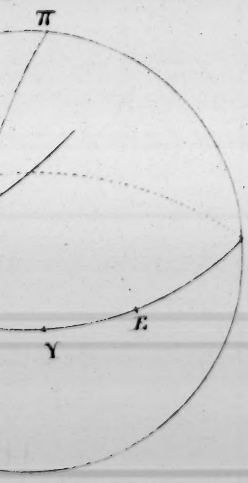
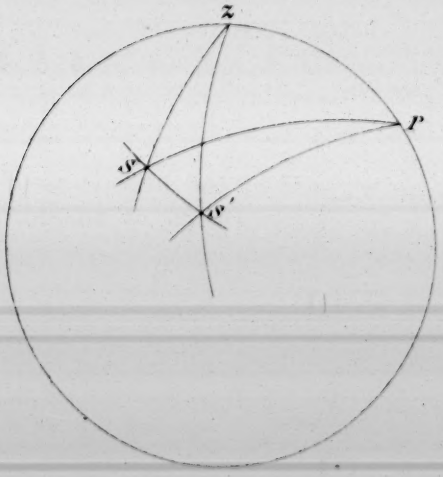



Fig. 5